SINGLE ITEM INVENTORY MODELS FOR BACKORDERS AND LOST SALES

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SUMMARY

The objective of this investigation is to study the single item inventory system in which during the stockout period, a fraction of the demand is backordered and the remaining demand is lost. The analysis of the system includes the formulation of the average annual cost model for the system and the subsequent minimization of this cost in order to obtain the optimal operating doctrine.

For the deterministic system where the demand is assumed to be constant, three types of models have been formulated. The first model is based on the assumption that during the stockout period the ratio of backorders to demand remains a constant. The second model assumes a linear increase of the backorders to demand ratio during the stockout period, whereas in the third model the ratio depends upon the time remaining for the stock to arrive and increases exponentially as the time for the arrival of stock approaches.

The study of the system with stochastic demands includes the formulation of two types of models. The first is a Lot Size Reorder Point model, or the (Q,r) model, in which a quantity Q is ordered each time the inventory level reaches the reorder level r. The second type is a Periodic Review model, or the (R,T) model, where at each review time a sufficient quantity is ordered to bring the inventory position up to a level R. It is assumed that the fraction of demand backordered during the stockout period is known and remains a constant throughout the stockout period.

In the real world inventory systems, such situations are generally encountered where during the stockout period some demands are backordered and some are lost. It is anticipated that the models presented in this study will be useful in the practical solutions of the aforementioned type of inventory problems.

CHAPTER I

INTRODUCTION

1.1 General Background

Inventory control has been, and no doubt will continue to be, one of the most important areas for operational analysis. Churchman et al (3) have suggested that more operation research effort has been directed towards the solution of inventory problems than toward any other area in business and industry. This has occurred because often a sizable portion of the assets of any manufacturing or distribution enterprise is tied up in inventories. According to Hadley and Whitin (7), in the United States the total dollar investment in inventories at any one time is immense. This investment represents more than 50 billion dollars for defence projects alone and more than 95 billion dollars for the private enterprise sector of the economy. This dollar level of investment in inventories necessitates the need for more efficient inventory control systems.

Effective inventory control in any organization regardless of how complicated the inventory system may be, requires the determination of how much inventory to carry. Enough must be carried so that the demands can be met, but not so much that the cost of inventory becomes excessive relative to demand. The optimal level of inventory is that which results in optimizing some appropriate measure of effectiveness for the system. Frequently the measure of effectiveness is either maximizing the profit or minimizing the cost. For example, in a retail store where the inventory of an item affects the revenue, it will be desirable to maximize the profit for the system whereas in a raw material inventory system, one would desire to minimize the system cost. It is, however, not necessary that maximizing the profit would yield the same results as minimizing the cost. To maintain the optimal level of inventory, a set of decision rules should be developed to indicate when to order and in what quantity. The development of the appropriate decision rules requires the following steps:

(i) determination of the properties of the system,

(ii) formulation of the inventory problem,

(iii) development of the model of the system and

(iv) derivation of a solution for the system.

It has been observed that because of high inventory carrying cost, the optimal inventory level obtained from these decision rules may not always result in a policy where the system is never permitted to have a stockout. Hadley and Whitin (7) comment that:

> It is seldom economically sound for an inventory system to carry enough inventory so there will always be stock on hand when a demand occurs. Because of the stochastic nature of the demand pattern, there can be times when demands occur and the system is out of stock. An important characteristic of the process generating demands is what happens when a demand occurs and the system is out of stock. Basically, there are two possibilities. Either the demand is lost or it is backordered... .

Much effort has been directed towards the formal modeling and analysis of inventory systems where stockouts are permitted. However, this analysis has been based on the assumption that during the stockout period either all the demands occurring are lost or all of them are backordered. Various mathematical models have been developed and analysed for the backorder case and the lost sales case, respectively (4), (5), (7) and (10). In some inventory systems, however, it is possible that when the system is out of stock, some demands occur and are backordered while others are lost. As an example, consider a department store where a particular item is out of stock and the new stock is expected to arrive in a few days. Of the customers for this item, who come to the store during the period the item is out of stock, some may place their orders with the store and wait until the stock arrives while some others may choose to buy the item elsewhere.

The specific aim of this investigation is to model and analyze the single item inventory system in which during the stockout period a fraction of demand is backordered and the remaining demand is lost. Both inventory models with deterministic demand and inventory models with stochastic demand will be considered.

The procedure used to achieve this objective will consist of:

1. Developing an appropriate model of the inventory system, and

2. Developing computational procedures for obtaining the optimal operating doctrine.

In addition, this research will discuss the relationship of these models to the usual backorders and lost sales models which, as will be seen, are special cases of the models formulated in this study.

1.2 Literature Survey

A considerable amount of research in inventory systems has been performed in the last few years. The important results have been summarized in several books. Fetter and Dalleck (5) have compiled a collection of inventory decision models. In their treatment of inventory systems where stockouts are permitted, they have considered the cases of demand lost and demand backordered during the stockout period separately.

Hadley and Whitin (7) have given an excellent account of inventory systems analysis. The authors have presented a variety of models for deterministic as well as stochastic situations. For the deterministic single item system, they have examined the cases where (a) all demands occurring when the system is out of stock are backordered, and (b) all demands occurring when the system is out of stock are lost. No attempt, however, has been made to develop a single model in which some demands during the stockout period are lost and some are backordered.

Naddor (10) presents various inventory models involving deterministic and stochastic processes. He has considered those inventory systems also where the demand rate is not constant but varies, and the nature of the variability is known.

All the references cited above base their models for the inventory systems where stockouts are permitted on the assumption that during the stockout period, either all the demands which occur are lost or all are backordered. However, Fabrycky and Banks (4) have formulated a single model for the backorders and lost sales case assuming deterministic demand, but no analysis of the model has been carried out to obtain the optimal operating procedure. Also, it is not obvious that the solution to their model will yield an absolute minimum. Furthermore, it was observed that minimizing the cost model that the authors had formulated is not equivalent to maximizing the profit, and therefore, one model cannot simultaneously accomplish the two objectives of the inventory analysis mentioned above. However, it shall be shown that by proper definition of the stockout cost, the minimization of cost will yield the same results as maximization of profit.

CHAPTER II

DETERMINISTIC SINGLE ITEM MODELS

2.1 Introduction

The models presented in this chapter deal with the analysis of a single item inventory system which is assumed to have the following properties:

1. The demand per unit time is deterministic, constant, and continuous.

2. The procurement lead time is a constant, independent of the demand rate and the quantity ordered.

3. The unit cost of the item is independent of the quantity ordered.

4. There is a fixed cost π for each unit of demand during the stockout period whether that unit is backordered or lost. In addition, there is a time dependent backorder cost $\bar{\pi}t$, which is proportional to the length of the time the backorder exists.

5. When a procurement arrives, all the backorders are met before the procurement quantity can be used to meet any other demands.

There are numerous ways in which the fraction of demand backordered may vary during the stockout period. For example, it is possible that this fraction remains constant throughout the stockout period. Alternatively it is also possible that there are proportionately more demands backordered as the time for the arrival of stock approaches. However, for the purposes of this study, the following three alternatives have been considered:

1. The ratio of backorders to demand remains constant during the stockout period.

2. The ratio of backorders to demand increases linearly during the stockout period.

3. The ratio of backorders to demand increases exponentially during the stockout period.

A model of average annual cost has been formulated and analyzed for each of the three cases above in order to obtain the optimal values of the decision variables. The following definitions and symbols will be used in the development of these average annual cost models:

D = the demand during the year.

T = the cycle time; that is, the time between the arrival of two consecutive orders.

Q = the order quantity.

t_1 = the length of time during the cycle when the system has stock.

t_2 = the length of time during the cycle when the system is out of stock.

C = the unit cost of an item.

I = the inventory carrying charge per year as a fraction of the item cost.

A = the fixed ordering cost per order.

S = the total demand per cycle during the stockout period.

q = the total number of backorders per cycle during the stockout period.

π = the fixed penalty cost per unit demand during the stockout period.

$\bar{\pi}$ = the shortage cost per unit period per backorder.

π_o = the profit per unit.

b = the ratio of total backorders to total demand during the stockout period per cycle. We see that b is equal to q/S.

Since the models developed herein will be based on the assumption that a fraction of the demand occurring when the system is out of stock is backordered and the remaining lost, the annual revenues received will depend on the length of time for which the system is out of stock, and hence on the operating doctrine. It would seem then, that one would not necessarily obtain the same operating policy for a model which attempts to minimize the total system cost as for one which attempts to maximize the total system profit. However, it shall be shown that careful definition of stockout cost will allow either a cost or profit formulation to yield identical results. Observe that the following relationship holds true:

P_c = Profit per cycle = Revenues received per cycle - Cost incurred per cycle.

The cost incurred per cycle consists of the following components:

Fixed shortage cost + Time dependent backorder cost + Ordering and carrying costs.

Hence, $P_c = Q\pi_o - (\pi S + \frac{\bar{\pi} q t_2}{2} + H)$

where H is ordering and inventory carrying cost per cycle.

Now since $Q = DT - (1-b)S$, one can see that

$$P_c = \pi_o[DT - (1-b)S] - \pi S - \frac{\bar{\pi}qt_2}{2} - H = DT\pi_o - (1-b)\pi_o S - \pi S - \frac{\bar{\pi}qt_2}{2} - H$$

Hence, average annual profit is

$$P = \frac{DT\pi_o}{T} - \frac{(1-b)\pi_o S}{T} - \frac{\pi S}{T} - \frac{\bar{\pi}qt_2}{2T} - \frac{H}{T}.$$

If we define the sum

$$\frac{(1-b)\pi_o S}{T} + \frac{\pi S}{T} + \frac{\bar{\pi}qt_2}{2T}$$

as the annual stockout cost, the average annual profit becomes

$$P = D\pi_o - K,$$

where K is the average annual cost consisting of the sum of the stockout cost, and the ordering and carrying costs. Now since $D\pi_o$ is independent of the operating doctrine, one can see that maximizing the average annual profit is the same as minimizing K.

2.2 A Deterministic Single Item Model with the Ratio of Backorders to Demand Remaining Constant During the Stockout Period

This section presents the formulation and analysis of a model for the case where the ratio of backorders to demand remains constant throughout the stockout period. The behavior of this system over time is shown in Figure 1. As can be seen from the Figure, the length of the cycle is

$$T = \frac{Q - q + S}{D} = \frac{Q - bS + S}{D} = \frac{Q + S(1-b)}{D}$$

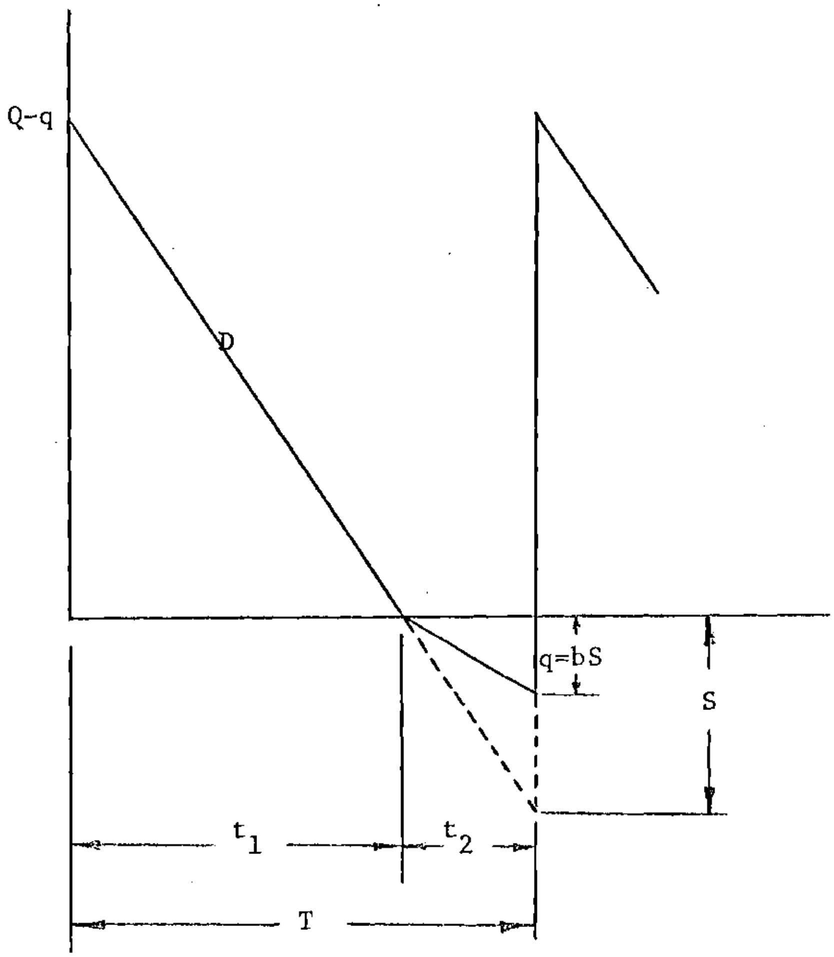

Figure 1. A Lot Size System with Ratio of Backorders to Demand Remaining Constant During the Stockout Period.

and the components of the cycle are

$$t_1 = \frac{Q - q}{D}$$

$$t_2 = \frac{S}{D}$$

Now the Ordering cost per year is just $\frac{A}{T}$, or

$$\frac{A\ D}{Q+S(1-b)} \quad .$$

The inventory carrying cost per cycle is given by

$$\frac{IC(Q-q)t_1}{2} = \frac{IC(Q - q)^2}{2D} \quad .$$

Hence annual inventory carrying cost is given by

$$\frac{IC(Q - q)^2}{2DT} = \frac{IC(Q - q)^2}{2\ [Q+S(1-b)]} \quad .$$

Also, the annual stockout cost is given by

$$\frac{\pi S}{T} + \frac{\bar{\pi} q t_2}{2T} + \frac{(1-b)\pi_o S}{T} \quad ,$$

which is equal to

$$\frac{\pi SD}{Q+S(1-b)} + \frac{\bar{\pi} b S^2}{2\ [Q+S(1-b)]} + \frac{(1-b)\pi_o SD}{[Q+S(1-b)]}$$

Thus the average annual cost which is the sum of ordering, carrying and stockout costs becomes

$$K(Q,S) = \frac{AD}{Q+S(1-b)} + \frac{IC(Q-bS)^2}{2\ [Q+S(1-b)]} + \frac{\pi SD}{Q+S(1-b)} + \frac{\bar{\pi} bS^2}{2\ [Q+S(1-b)]}$$

$$+ \frac{(1-b)\pi_o SD}{Q+S(1-b)} \tag{1}$$

In order to find the optimal values of Q and S, say Q* and S*, which minimize the average annual cost, a necessary condition is that Q* and S* satisfy

$$\frac{\partial K}{\partial Q} = 0$$

$$\frac{\partial K}{\partial S} = 0$$

Differentiating (1) with respect to Q and equating the result to zero gives

$$\frac{\partial K}{\partial Q} = 0 = \left[\frac{1}{[Q+S(1-b)]^2}\right] \left[[Q + S(1-b)] \quad [IC(Q-Sb)] \right.$$
$$\left. - \left(AD + \frac{IC(Q-Sb)^2}{2} + \pi SD + \frac{\bar{\pi} bS^2}{2} + \pi_o(1-b)SD \right) \right]$$

simplication of which results in

$$\left[\frac{1}{[Q+S(1-b)]^2}\right] \left[\frac{ICQ^2}{2} + QSIC(1-b) + S^2\left(-ICb + \frac{ICb^2}{2} - \frac{\bar{\pi} b}{2} \right) \right.$$
$$\left. + S[\pi D + \pi_o D(1-b)] - AD \right] = 0 \tag{2}$$

Similarily,

$$\frac{\partial K}{\partial S} = 0 = \left[\frac{1}{[Q+S(1-b)]^2}\right]\left[[Q+S(1-b)]\ [-ICb(Q-Sb) + \pi D + \bar{\pi}bS + \pi_o(1-b)S]\right.$$

$$\left. - (1-b)\left(AD + \frac{IC(Q-Sb)^2}{2} + \pi SD + \frac{\bar{\pi}bS^2}{2} + \pi_o(1-b)SD\right)\right],$$

simplication of which results in

$$\left[\frac{1}{[Q+S(1-b)]^2}\right]\left[Q^2\left(-\frac{ICb}{2} - \frac{IC}{2}\right) + QS(ICb^2 + \bar{\pi}b) + Q[\pi D+\pi_o D(1-b)]\right.$$

$$\left. + S^2\left(\frac{ICb^2(1-b)}{2} + \frac{\bar{\pi}b(1-b)}{2}\right) - (1-b)A\ D\right] = 0 \tag{3}$$

It can be seen that because of the complexity of (2) and (3) it is not possible to obtain Q* and S* directly in terms of other system parameters by solving the two equations simultaneously.

Consider the transformation

$$U = Q + S(1-b)\ , \tag{4}$$

$$\text{and}\quad V = Q - Sb. \tag{5}$$

therefore one can see that $U - V = S$ and $V + (U-V)b = Q$. Referring to Figure 2, U can be defined as the total demand during the cycle and V can be defined as the on hand inventory at the beginning of the cycle.

The average annual cost given by (1) can be written in terms of U and V as

$$K(U,V) = \frac{AD}{U} + \frac{ICV^2}{2U} + \frac{\pi(U-V)D}{U} + \frac{\bar{\pi}b(U-V)^2}{2U} + \frac{\pi_o(1-b)(U-V)D}{U}\ . \tag{6}$$

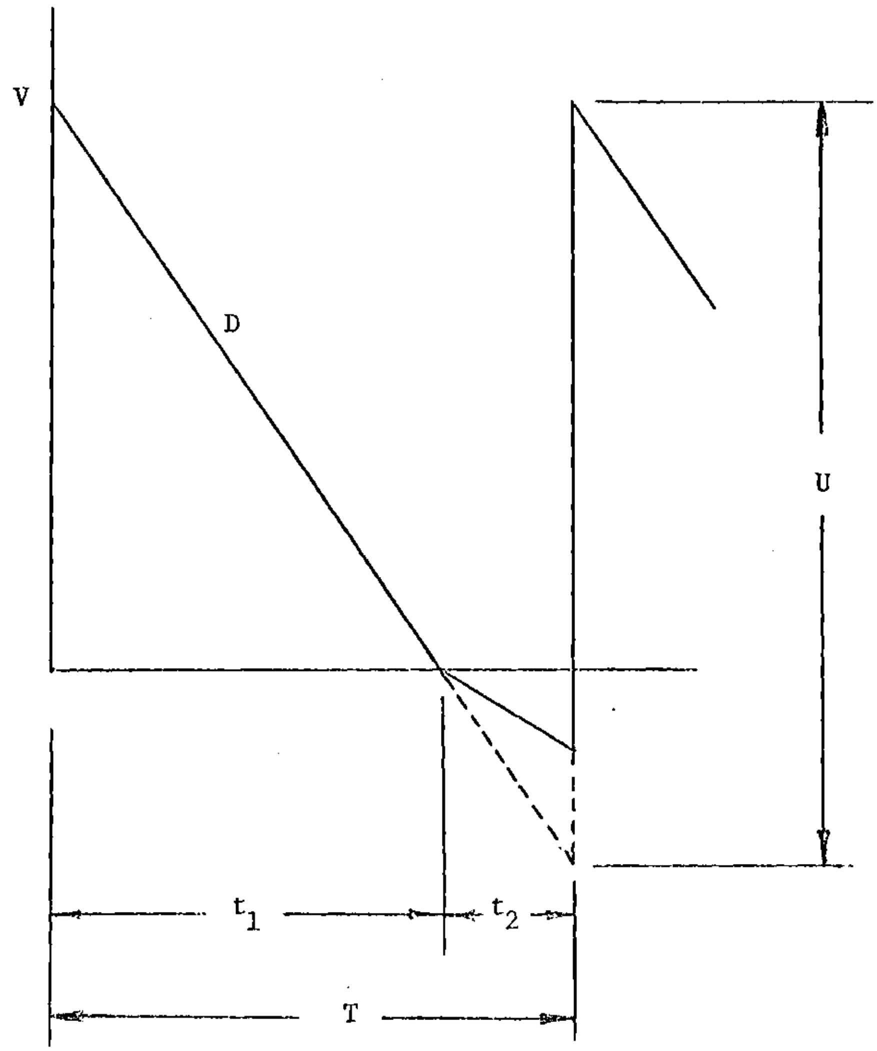

Figure 2. A Lot Size System with Ratio of Backorders to Demand Remaining Constant During the Stockout Period - Alternate View.

The solution procedure will now be to first obtain an optimal solution to (6) and then convert it into an optimal solution for (1) by using the relationships (4) and (5). Therefore, it is essential to prove that the optimal solution to (6) corresponds to the optimal solution to (1). The proof is easily established. From an earlier statement, average annual profit can be written as

$$P = \pi_o D - K(Q,S) \quad . \tag{7}$$

where K(Q,S) is given by (1). Now using the relationships (4) and (5)

$$P = \pi_o D - K(U,V) \quad . \tag{8}$$

Since $\pi_o D$ is a constant, K(Q,S) will be minimum if $\pi_o D - P$ is minimum and $\pi_o D - P$ will also be minimum if K(U,V) is minimum. Therefore, one can see that (1) is minimum when (6) is minimum, or the optimal solution to (6) corresponds to the optimal solution to (1).

K(U,V) is a function of U and V. We wish to find the absolute minimum of K(U,V) in the region $0<U<\infty$ and $0<V<\infty$. For any finite value of V, $0<V<\infty$, K(U,V) is infinite when $U = 0$ or $U = \infty$. Thus the optimal value of U must satisfy $0<U^*<\infty$. For the optimal value of V to satisfy $0<V<\infty$, U^*, V^* must satisfy

$$\frac{\partial K}{\partial V} = 0$$

$$\frac{\partial K}{\partial U} = 0 \quad .$$

Differentiating (6) with respect to V and equating the result to zero gives

$$\frac{\partial K}{\partial V} = 0 = \frac{1}{U}\left(ICV - \pi D - \bar{\pi}b(U-V) - \pi_o D(1-b)\right) ,$$

or

$$\frac{1}{U}\left(V(IC + \bar{\pi}b) - [\pi D + \pi_o(1-b)] - \bar{\pi}bU\right) = 0 .$$

Putting $IC + \bar{\pi}b = B$ and $\pi D + \pi_o(1-b) = R$,

$$\frac{\partial K}{\partial V} = 0 = \frac{1}{U}\ (BV - R - \bar{\pi}\ bU) .$$ Therefore

$$\bar{\pi}\ bU = BV - R \tag{9}$$

Also, in a similar fashion one may obtain

$$\frac{\partial K}{\partial U} = 0 = \frac{1}{U^2}\left[U[\pi D + \bar{\pi}b(U-V) + \pi_o(1-b)D] - \left(AD + \frac{ICV^2}{2} + \pi(U-V)D + \frac{\bar{\pi}b(U-V)^2}{2} + \pi_o(1-b)(U-V)D\right)\right].$$

Simplification of the above results in

$$\frac{\partial K}{\partial U} = 0 = \frac{1}{U^2}\left[U\bar{\pi}b(U-V) - \left(AD + \frac{ICV^2}{2} - \pi DV + \frac{\bar{\pi}b(U^2+V^2-2UV)}{2} + \pi_o(1-b)DV\right)\right]$$

$$= \frac{1}{U^2}\left[\frac{\bar{\pi}bU^2}{2} - V^2\left(\frac{IC}{2} + \frac{\bar{\pi}b}{2}\right) + V\ [\pi D + \pi_o(1-b)D] - AD\right]$$

$$= \frac{1}{U^2} \left[\frac{\bar{\pi} b U^2}{2} - \frac{BV^2}{2} + RV - AD \right] .$$

Hence $\dfrac{\bar{\pi} b U^2}{2} = \dfrac{BV^2}{2} - RV + AD$ (10)

The second partial derivatives are

$$\frac{\partial^2 K}{\partial V^2} = \frac{B}{U} \quad , \tag{11}$$

$$\frac{\partial^2 K}{\partial V \ \partial U} = \frac{U(-\bar{\pi} b) - (BV - R - \bar{\pi} b U)}{U^2} \quad ,$$

$$= - \frac{(BV-R)}{U^2} \quad .$$

But from (9) one can have

$$\bar{\pi} b U = BV - R,$$

hence

$$\frac{\partial^2 K}{\partial V \ \partial U} = \frac{\bar{\pi} b U}{U^2} = - \frac{\bar{\pi} b}{U} \quad . \tag{12}$$

Also,

$$\frac{\partial^2 K}{\partial U^2} = \frac{1}{U^4} \left[U^2 (U \bar{\pi} b) - 2U \left(\frac{U^2 \bar{\pi} b}{2} - \frac{BV^2}{2} + RV - AD \right) \right]$$

or

$$= \frac{1}{U^4} \left[2U \left(\frac{BV^2}{2} - RV + AD \right) \right] \quad .$$

One may obtain from (10)

$$\frac{\bar{\pi} b U^2}{2} = \frac{BV^2}{2} - RV + AD \quad ,$$

hence

$$\frac{\partial^2 K}{\partial U^2} = \frac{\bar{\pi} b U^3}{U^4} = \frac{\bar{\pi} b}{U} \quad . \tag{13}$$

Now a function in n variables $f(x_1, x_2, \ldots, x_n)$ is said to be convex if and only if its (n x n) Hessian matrix is positive semidefinite for all possible values of $(x_1, x_2, \ldots, x_n)$. For example, if there are two variables, then $f(x_1, x_2)$ is convex if and only if

$$\begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} \\[2ex] \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \dfrac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

is positive semidefinite for all values of (x_1, x_2). One can see that this matrix will be positive semidefinite if and only if

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} \; \frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} - \left[\frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2} \right]^2 \geq 0 \quad ,$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} \geq 0 \quad ,$$

and

$$\frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} \geqslant 0 .$$

Therefore, the necessary and sufficient conditions for K(U,V) to be convex are just

$$\frac{\partial^2 K}{\partial U^2} \frac{\partial^2 K}{\partial V^2} - \left[\frac{\partial^2 K}{\partial V \partial U} \right]^2 \geqslant 0 ,$$

$$\frac{\partial^2 K}{\partial U^2} \geqslant 0 ,$$

and $$\frac{\partial^2 K}{\partial V^2} \geqslant 0 .$$

It is evident from (11), (12) and (13) that for any positive finite value of U, the above conditions will always hold and hence, it can be concluded that K(U,V) is a convex function. Therefore, U* and V* will yield an absolute minimum.

Solving (9) and (10) one can obtain

$$U^* = \sqrt{\frac{2ADB - R^2}{IC\bar{\pi}b}} , \tag{14}$$

and $$V^* = \frac{RIC + \sqrt{IC\bar{\pi}b(2ADB - R^2)}}{BIC} . \tag{15}$$

The implications of this solution are not obvious, and in fact, three specific situations must be considered. The analysis of the solution for the case $\bar{\pi} \neq 0$ is presented first.

Case 1 $\quad 2ADB < R^2$

There are no real values of U and V satisfying (9) and (10), there is no solution V, $0<V<\infty$ which yields a minimum of K(U,V) and hence the solution lies on the boundaries so that V = 0 or V = ∞. The solution obviously is V = 0 because V = ∞ means that the on hand inventory at the beginning of the cycle is ∞. This would mean that the carrying costs, and thus K(U,V), is ∞.

Therefore, V = 0, and from (10) with V = 0

$$U = \sqrt{\frac{2AD}{\bar{\pi}b}} \quad ,$$

and K(U,V) from (6) becomes

$$K(U,V) = \sqrt{2\bar{\pi}bAD} + \pi D + \sqrt{\frac{AD\bar{\pi}b}{2}} + \pi_o(1-b)D \tag{16}$$

The optimal solution can be obtained by comparing K(U,V) with K_w which is the average annual cost under the optimal policy when no stockouts are allowed and is equal to $\sqrt{2ADIC}$. If $K(U,V) < K_w$ then the optimal solution is

$$V^* = 0, \quad U^* = \sqrt{\frac{2AD}{\bar{\pi}b}} \quad , \quad K^* = K(U,V).$$

If $K(U,V) > K_w$, then the optimal solution is

$$U^* = V^* = \sqrt{\frac{2AD}{I\ C}}\ , \quad \text{and } K^* = K_w\ .$$

Case 2 $\quad 2ADB = R^2$

From (14) and (15)

$$V = \frac{R}{B} \text{ and } U = 0\ .$$

Since for any finite value of V, when $U = 0$, $K(U,V) = \infty$, hence the optimal solution is not to allow stockouts, and

$$U^* = V^* = \sqrt{\frac{2AD}{I\ C}}\ , \quad K^* = K_w\ .$$

Case 3 $\quad 2ADB > R^2$

If $U>V$, the solution is optimal provided that $K(U,V) < K_w$. In the case of $K(U,V) > K_w$, the optimal solution will be

$$U^* = V^* = \sqrt{\frac{2AD}{I\ C}}\ , \quad \text{and} \quad K^* = K_w\ .$$

If $U < V$, it means that the on hand inventory at the beginning of the cycle is greater than the demand during the cycle, or at the end of the cycle there will remain some safety stock. Therefore, $K(U,V)$ will always be greater than K_w and the optimal solution will be

$$U^* = V^* = \sqrt{\frac{2AD}{I\ C}}\ , \quad \text{and} \quad K^* = K_w\ .$$

An analysis of results is now presented for the case when $\bar{\pi} = 0$. When $\bar{\pi} = 0$, from (9) and (10)

$$BV - R = 0$$

and $$BV^2 - 2RV + 2AD = 0$$

Eliminating V, one obtains $R^2 = 2ABD$ which is not true in general. That means when $\bar{\pi} = 0$, there is no solution V, $0 < V < \infty$, and hence the solution lies on boundaries so that $V = 0$ or $V = \infty$. Again the solution is $V = 0$ because $V = \infty$ means that the on hand inventory at the beginning of the cycle is ∞ which implies that carrying costs and hence K(U,V) is ∞. Therefore, $V = 0$ and K(U,V) from (6) becomes

$$K(U,V) = \frac{A\ D}{U} + \pi D + \pi_o(1-b)D$$

Obviously K(U,V) will be minimum when $U = \infty$, and therefore, when $V = 0$ and $U = \infty$,

$$K(U,V) = \pi D + \pi_o(1-b)D$$

The optimal solution can be obtained by comparing K(U,V) with K_w. If $K(U,V) < K_w$, the optimal solution is

$$U^* = \infty, \quad V^* = 0 \quad \text{and} \quad K^* = K(U,V)\ .$$

If $K(U,V) > K_w$, the optimal solution is

$$U^* = V^* = \sqrt{\frac{2AD}{I\ C}}\ , \quad \text{and } K^* = K_w.$$

The above ideas may be incorporated in a formal algorithm to obtain the optimal solution.

This algorithm is as follows:

Step 1 : If $\bar{\pi} = 0$, go to Step 9. Otherwise proceed to Step 2.

Step 2 : Compute the value of 2ADB and R^2 and proceed to Step 3.

Step 3 : If $2ADB < R^2$, go to Step 6, otherwise go to Step 4.

Step 4 : If $2ADB = R^2$, go to Step 7, otherwise go to Step 5.

Step 5 : If $2ADB > R^2$, go to Step 8.

Step 6 : Compute the value of U given by

$$U = \sqrt{\frac{2AD}{\bar{\pi}b}} \ ,$$

use this value of U and V = 0 to find the value of K(U,V) from (6). Then go to Step 10.

Step 7 : The optimal solution is given by

$$U^* = V^* = \sqrt{\frac{2AD}{I\ C}} \ , \text{ and } K^* = K_w \ .$$

Terminate.

Step 8 : Compute the values of U and V from (14) and (15) respectively. Use these values of U and V in (6) to compute the value of K(U,V). Then go to Step 10.

Step 9 : Compute the value of K(U,V) from (6) with V = 0, and $U = \infty$. Proceed to Step 10.

Step 10: Compute the value of K_w . If $K(U,V) < K_w$, K(U,V) is optimal. Otherwise K_w is optimal. Terminate.

The procedure is illustrated in the following numerical examples.

Example 2.1* Consider an item with the following characteristics:

D = 200 units per year, C = \$25

I = 0.20 A = \$5

π = \$0.20 per unit, $\bar{\pi}$ = \$10 per unit per year.

π_o = \$5 b = 1.0

Then

$$B = IC + \bar{\pi}b \ ,$$

$$= (0.20)(25) + (10)(1.0) = 15$$

$$R = D + \pi_o(1-b)$$

$$= (0.20)(200) + (5)(1-1) = 40$$

$$2ADB = (2)(5)(200)(15) = 30000$$

$$R^2 = (40)^2 = 1600 \ .$$

Since $2ADB > R^2$, therefore, from (14)

$$U = \sqrt{\frac{2ADB - R^2}{IC\bar{\pi}b}} = \sqrt{\frac{30000 - 1600}{(.20)(25)(10)(1.0)}}$$

$$= 23.8$$

Then from (15) $$V = \frac{R}{B} + \frac{\sqrt{IC\bar{\pi}b(2ADB - R^2)}}{BIC}$$

or $$V = \frac{40}{15} + \frac{\sqrt{(.20)(25)(10)(1.0)(30000-1600)}}{(15)(.20)(25)}$$

$$= 18.53$$

* Since b = 1, all the demands during stockout period are backordered. The problem therefore is the same as the one solved in (7) p.47.

From (6)

$$K(U,V) = \frac{1}{23.8}\left[(5)(200) + \frac{(.20)(25)(18.53)^2}{2} + (.20)(23.8 - 18.53)(200)\right.$$

$$\left. + \frac{(10)(1)(23.8 - 18.53)^2}{2} + (5)(1-1)(23.8 - 18.53)(200)\right]$$

$$= 84.0 \quad ,$$

and $K_w = \sqrt{2ADIC} = \sqrt{2(5)(200)(.20)(25)}$

$$= 100$$

Since $K(U,V) < K_w$, the optimal solution is

$$U^* = 23.8 \;, \quad V^* = 18.53$$

or from (4) and (5)

$$Q^* = 23.8 \;, \quad S^* = 5.27$$

and $\quad K^* = 84.0$.

This is the identical solution found by Hadley and Whitin.

<u>Example 2.2</u> In the Example 2.1, by changing the value of b from 1.0 to 0.6 one obtains

$$B = (.20)(25) + (10)(.6) = 11.0$$

$$R = (.20)(200) + (5)(1 - .6) = 42.0$$

$$2ADB = (2)(5)(200)(11.0) = 22000$$

$$R^2 = (42.0)^2 = 1764 \quad .$$

Since $2ADB > R^2$, one obtains from (14)

$$U = \sqrt{\frac{22000 - 1764}{(.20)(25)(10)(.6)}}$$

$$= 25.95 \ .$$

From (15)

$$V = \frac{42.0}{11.0} + \frac{\sqrt{(.20)(25)(10)(.6)(22000 - 1764)}}{(11.0)(.20)(25)}$$

$$= 17.92 \ .$$

From (6)

$$K(U,V) = \frac{1}{25.95} \left[(5)(200) + \frac{(.20)(25)(25.95 - 17.92)^2}{2} \right.$$

$$+ (.20)(25.95 - 17.92)(200) + \frac{(10)(25.95-17.92)^2(.6)}{2}$$

$$\left. + (5)(.6)(25.95 - 17.92)(200) \right]$$

$$= 213 \ .$$

But $K_w = 100$,

or $K_w < K(U,V)$

Therefore, the optimal solution is

$$U^* = V^* = \sqrt{\frac{2AD}{I\ C}} = \sqrt{\frac{2(5)(200)}{(.20)(25)}} = 20$$

or $Q^* = 20$ and $S^* = 0$

<u>Example 2.3</u> Consider an item with the following characteristics:

$D = 200$, $C = \$25$

$I = 0.60$, $A = \$15$

$\pi = 0.10$, $\qquad \bar{\pi} = \$1$

$\pi_o = \$4$ $\qquad b = 0.8$

One may obtain

$$B = (.60)(25) + (1)(.8) = 15.8 \ .$$

$$R = (.10)(200) + (4)(1 - .8) = 20.8 \ .$$

$$2ADB = 2(15)(200)(15.8) = 94800 \ ,$$

and $\quad R^2 = (20.8)^2 \qquad = 433 \ .$

Since $2ADB > R^2$, one can have from (14)

$$U = 88.6 \ ,$$

and from (15)

$$V = 5.78 \ .$$

Also, from (6)

$$K(U,V) = 235.74 \ .$$

But $\quad K_w = \sqrt{2(15)(.6)(200)(25)} = 300 \ ,$

and since $K(U,V) < K_w$, the optimal solution is

$$U^* = 88.6 \ , \quad V^* = 5.78 \ , \quad \text{and } K^* = 235.74 \ .$$

Or $\quad Q^* = 72.03$, and $S^* = 82.82$.

<u>Example 2.4</u> Consider an item with the following characteristics:

$D = 200$, $\qquad C = \$25$

$I = 0.20$, $\qquad A = \$5$

$\pi = \$0.75$ $\qquad \bar{\pi} = \$10$

$\pi_o = \$5$ $\qquad b = 0.6$

$$B = (.20)(25) + (10)(.6) = 11 \ .$$

$$R = (.75)(200) + (5)(1 - .6) = 152 \ .$$

Hence $2ADB = 22000$,

and $R^2 = 23100$.

From the above one can see that $2ADB < R^2$, hence

$$V = 0 ,$$

and $$U = \sqrt{\frac{2AD}{\bar{\pi} b}} = \sqrt{\frac{2(5)(200)}{(10)(.6)}} .$$

From (16) it is obvious that $K(U,V) >> \pi D = 150$. But $K_w = 100$,

therefore, $K_w < K(U,V)$, and the optimal solution is

$$U^* = V^* = \sqrt{\frac{2AD}{I\ C}} = \sqrt{\frac{2(5)(200)}{(.20)(25)}} = 25 ,$$

or $Q^* = 20$, and $S^* = 0$.

<u>Example 2.5</u> Consider an item with the following characteristics:

$D = 200$,	$C = \$25$,
$I = 0.20$,	$A = \$5$,
$\pi = \$0.20$,	$\bar{\pi} = 0$,
$\pi_o = \$2$,	$b = 0.6$.

When $\bar{\pi} = 0$, $V = 0$ and $U = \infty$,

and $$K(U,V) = \pi D + \pi_o(1-b)D$$

$$= (.20)(200) + (2)(1 - .6)(200)$$

$$= 200 .$$

But $K_w = \sqrt{2AIDC} = 100$, and since $K_w < K(U,V)$, the optimal solution is

$$U^* = V^* = \sqrt{\frac{2AD}{I\ C}} = 20 .$$

Or $Q^* = 20$, and $S^* = 0$.

Example 2.6 Make the following changes in the previous example:

$$A = \$10 , \quad D = 50 .$$

Now with $V = 0$ and $U = \infty$,

$$K(U,V) = \pi D + \pi_o(1-b)D$$

$$= (.20)(50) + (2)(1 - .6)(50)$$

$$= 50 .$$

And $K_w = \sqrt{(2)(10)(50)(.20)(25)}$

$$= 50\sqrt{2} .$$

Since $K_w > K(U,V)$, the optimal solution is

$$U^* = \infty, \quad \text{and } V^* = 0 .$$

Or $S^* = U^* - V^* = \infty$, which means the system should not be operated at all.

2.3 A Deterministic Single Item Model with the Ratio of Backorders to Demand Increasing Linearly During the Stockout Period

This model will be developed on the assumption that at the point in time when the system runs out of stock, the ratio of backorders to demand is known and is equal to p. This ratio increases linearly during the stockout period and at the end of the cycle, when the stock is about to arrive, the ratio equals 1. This is illustrated graphically in Figure 3. That means that no demand will be lost at the point in time when the stock is about to arrive.

From Figure 4 one can obtain

$$t_1 = \frac{Q-q}{D} ,$$

$$t_2 = \frac{S}{D} ,$$

and $$T = \frac{Q + S(1-b)}{D}$$

The annual ordering cost $= \dfrac{A}{T} = \dfrac{AD}{Q+S(1-b)}$.

The inventory carrying cost per cycle is

$$\frac{IC(Q-q)t_1}{2} = \frac{IC(Q-q)^2}{2D} .$$

Hence the annual inventory carrying cost is given by

$$\frac{IC(Q-q)^2}{2DT} = \frac{IC(Q-bS)^2}{2[Q+S(1-b)]} .$$

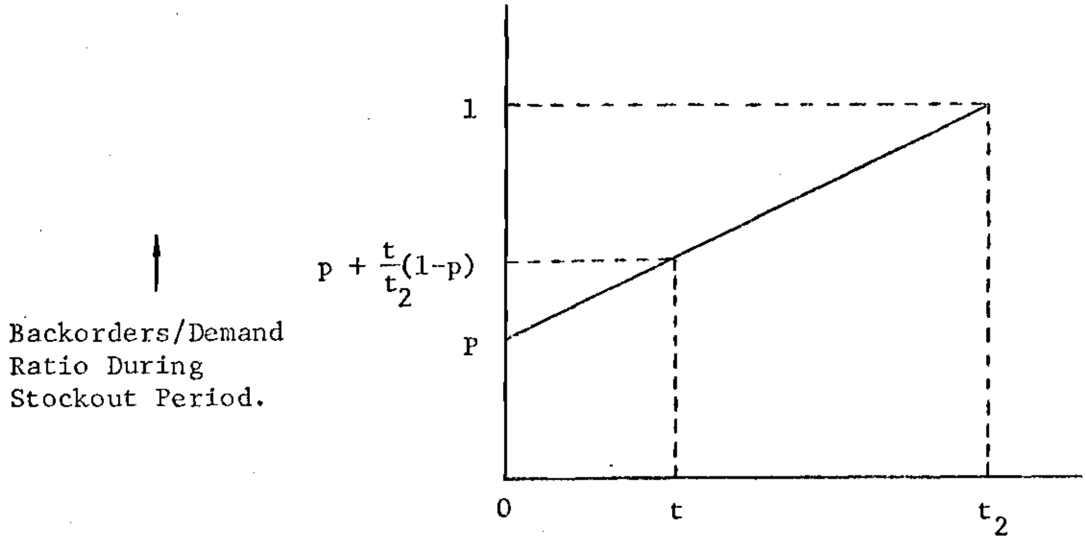

Figure 3. Linear Increase of Backorders to Demand Ratio During the Stockout Period.

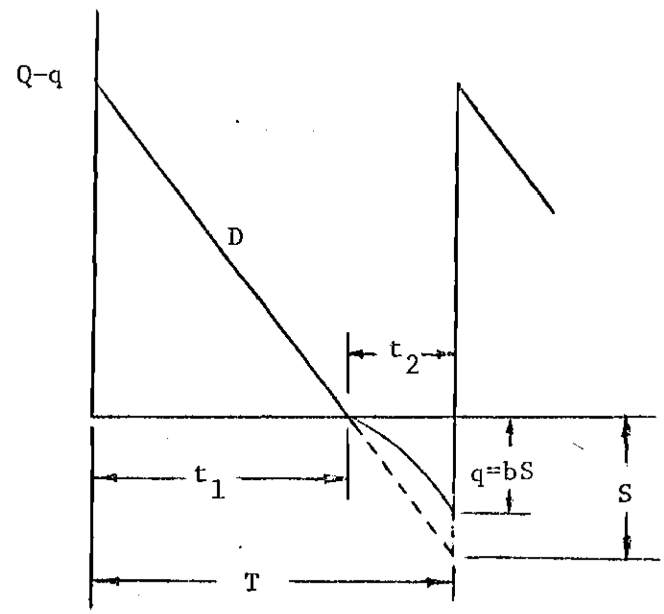

Figure 4. A Lot Size System with Ratio of Backorders to Demand Increasing During the Stockout Period.

As mentioned earlier, stockout cost is the sum of fixed shortage cost, the time dependent backorders cost and the lost profit. One can see that

Fixed Shortage cost per year $= \frac{\pi S}{T} = \frac{\pi SD}{Q+S(1-b)}$.

Referring to Figure 3, one can see that the backorders up to any time t during the stockout period are given by

$$\int_o^t D \left(p + \frac{t(1-p)}{t_2} \right) dt$$

or

$$= Dpt + \frac{D(1-p)t^2}{2t_2} .$$

Hence the time dependent backorder cost per cycle is

$$= \bar{\pi} \cdot \int_o^{t_2} \left(Dpt + \frac{D(1-p)t^2}{2t_2} \right) dt$$

$$= \frac{\bar{\pi} Dpt_2^2}{2} + \frac{\bar{\pi} D(1-p)t_2^2}{6} .$$

The annual time dependent backorder cost is

$$= \frac{1}{T} \left[\frac{\bar{\pi} Dpt_2^2}{2} + \frac{\bar{\pi} D(1-p)t_2^2}{6} \right]$$

$$= \frac{\bar{\pi} pS^2}{2[Q+S(1-b)]} + \frac{\bar{\pi}(1-p)S^2}{6[Q+S(1-b)]} \tag{17}$$

Now the total backorders during the stockout period are given by

$$q = \int_o^{t_2} D \left(p + \frac{t(1-p)}{t_2} \right) dt$$

$$= Dpt_2 + \frac{D(1-p)t_2}{2}$$

Hence the lost sales during the stockout period are

$$S - q = Dt_2 - Dpt_2 - \frac{D(1-p)t_2}{2}$$

$$= \frac{(1-p)Dt_2}{2} \quad .$$

The lost profit per cycle is $\quad \dfrac{\pi_o(1-p)Dt_2}{2} \quad ,$

and therefore, annual lost profit is $\quad \dfrac{\pi_o(1-p)Dt_2}{2T} \quad ,$

or $\quad \dfrac{\pi_o(1-p)SD}{2[Q+S(1-b)]} \quad .$

Now $\quad b = \dfrac{q}{S}$

$$= \left[Dpt_2 + \frac{D(1-p)t_2}{2}\right] \left[\frac{1}{Dt_2}\right]$$

$$= \frac{p+1}{2} \quad ,$$

or $\quad p = 2b - 1 \quad .$

Substituting this value of p in (17) and (18), the annual time dependent backorder cost is

$$\frac{\bar{\pi}S^2}{Q+S(1-b)} \left(\frac{2b-1}{2} + \frac{2-2b}{6} \right)$$

$$= \frac{\bar{\pi}(4b-1)S^2}{6[Q+S(1-b)]} \quad ,$$

and the annual lost profit is

$$\frac{\pi_o(1-b)SD}{Q+S(1-b)} \quad .$$

The average annual cost is then given by

$$K(Q,S) = \frac{AD}{Q+S(1-b)} + \frac{IC(Q-bS)^2}{2[Q+S(1-b)]} + \frac{\pi SD}{Q+S(1-b)} + \frac{\bar{\pi}(4b-1)S^2}{6[Q+S(1-b)]}$$

$$+ \frac{\pi_o(1-b)SD}{Q+S(1-b)} \quad . \tag{19}$$

It is interesting to note that the only difference between the total costs given by (1) and (19) is due to the difference in the time dependent backorder costs.

Again letting $U = Q+S(1-b)$

and $V = Q-Sb$,

the transformed cost function becomes

$$K(U,V) = \frac{AD}{U} + \frac{ICV^2}{2U} + \frac{\pi(U-V)D}{U} + \frac{\bar{\pi}(4b-1)(U-V)^2}{6U}$$

$$+ \frac{\pi_o D(1-b)(U-V)}{U} \quad . \tag{20}$$

A necessary condition that U* and V* be optimal is that they satisfy

$$\frac{\partial K}{\partial U} = 0 \quad ,$$

$$\frac{\partial K}{\partial V} = 0 \quad .$$

Taking these derivatives and equating the results to zero, one obtains

$$\frac{\partial K}{\partial V} = 0 = \frac{1}{U} \left[ICV - \pi D - \frac{\bar{\pi}(4b-1)(U-V)}{3} - \pi_o D(1-b) \right]$$

$$= \frac{1}{U} \left[V \left(IC + \frac{\bar{\pi}(4b-1)}{3} \right) - [\pi D + \pi_o(1-b)] - \frac{\bar{\pi}(4b-1)U}{3} \right] \quad .$$

Letting $IC + \dfrac{\bar{\pi}(4b-1)}{3} = B'$,

$\pi D + \pi_o(1-b) = R$,

and $\dfrac{4b-1}{3} = b'$,

$$\frac{\partial K}{\partial V} = 0 = \frac{1}{U} (B'V - R - \bar{\pi}b'U)$$

or

$$\bar{\pi}b'U = B'V - R \quad . \qquad (21)$$

Also, $$\frac{\partial K}{\partial U} = 0 = \frac{1}{U^2} \left[U \left(\pi D + \frac{\bar{\pi}(4b-1)(U-V)}{3} + \pi_o(1-b)D \right) - \left(AD + \frac{ICV^2}{2} + \pi(U-V)D + \frac{\bar{\pi}(4b-1)(U-V)^2}{6} + \pi_o D(1-b)(U-V) \right) \right] \quad .$$

Simplification of the above results in

$$\frac{\partial K}{\partial U} = 0 = \frac{1}{U^2} \left[\frac{\bar{\pi}(4b-1)U^2}{6} - V^2 \left(\frac{IC}{2} + \frac{\bar{\pi}(4b-1)}{6} \right) + V[\pi D + \pi_o(1-b)] - AD \right] \quad ,$$

$$\text{or} \quad \frac{\bar{\pi}b'U^2}{2} - \frac{B'V^2}{2} + RV - AD = 0 \quad ,$$

$$\text{or} \quad \frac{\bar{\pi}b'U^2}{2} = \frac{B'V^2}{2} - RV + AD \quad . \tag{22}$$

Because of the analogy between (21), (22) and (9), (10) respectively, the results obtained and their analysis will be similar to the results presented in section 2.1 of this chapter for the deterministic model having a constant ratio of backorders to demand during the stockout period. Therefore, the solution to (20) is given by

$$U = \sqrt{\frac{2ADB' - R^2}{IC\bar{\pi}b'}} \quad , \tag{23}$$

$$\text{and} \qquad V = \frac{RIC + \sqrt{IC\bar{\pi}b\ (2ADB' - R^2)}}{B'IC} \quad , \tag{24}$$

which will always yield an absolute minimum.

Example 2.7 Consider an item with the following characteristics:

$D = 200$	$C = \$25$
$I = 0.20$	$A = \$5$
$\pi = \$0.20$	$\bar{\pi} = \$10$
$\pi_o = \$2$	$p = 0.8$

One may compute

$$b = \frac{p+1}{2} = 0.9$$

$$b' = \frac{4b-1}{3} = \frac{2.6}{3} = .866$$

$$B' = IC + \frac{\bar{\pi}(4b-1)}{3}$$

$$= (.20)(25) + (10)(.866)$$

$$= 13.66 \quad .$$

$$R = \pi D + \pi_o(1-b)$$

$$= (.20)(200) + (2)(.1)$$

$$= 40.2 \quad .$$

Also, $2ADB' = (2)(5)(200)(13.66) = 27320$

and $R^2 = (40.2)^2 = 1620 \quad .$

Now since $2ADB' > R^2$, one obtains from (23)

$$U = \sqrt{\frac{27320 - 1600}{(.20)(25)(10)(.866)}} \quad ,$$

$$= 24.40 \quad ,$$

from (24),

$$V = \frac{40.2}{13.66} + \frac{\sqrt{(.20)(25)(10)(.866)(27320 - 1620)}}{(13.66)(.20)(25)} \quad ,$$

$$= 18.40 \quad ,$$

and from (20),

$$K(U,V) = 103.5 \quad .$$

But $K_w = \sqrt{2ADIC} = \sqrt{2(5)(200)(.20)(25)}$

$$= 100 \quad ,$$

and since $K_w < K(U,V)$, the optimal solution is

$$U^* = V^* = \sqrt{\frac{2AD}{I\,C}} = \sqrt{\frac{2(5)(200)}{(.20)(25)}} = 20 \quad ,$$

or $Q^* = 20$ and $S^* = 0$.

In order to observe the effect of p on K(U,V), the relationship between p and K(U,V) is shown as Figure 5. It will be seen that for an item with characteristics as given in example 2.7, K(U,V) will always be greater than K_w if p is less than p_o. Hence for all values of p less than p_o, the optimal policy would be not to allow any stockout. In general, for any inventory system for which the system parameters are known, the relationship in Figure 5 can be obtained and value of p_o which corresponds to the system cost K_w can be found. The optimal policy is then easy to determine depending upon whether the value of p for the inventory system is greater than or less than p_o.

2.4 A Deterministic Single Item Model with the Ratio of Backorders to Demand Increasing Exponentially During the Stockout Period

This model will be developed using the assumption that at a point in time, say t, during the stockout period, the ratio of backorders to demand is given by $e^{-\frac{t_2-t}{N}}$. A graphical illustration is given in Figure 6. In other words, the ratio depends upon (t_2-t), the time period remaining for the order quantity to arrive and the greater this time period the smaller the ratio will be. The desired nature of the curve between this ratio and the time remaining for the order quantity to arrive will determine the value of N. For example, with N = 0.83:

when $t_2-t = 0$, the fraction of demand backordered is 1,

when $t_2-t = \frac{5}{12}$, the fraction of demand backordered is 0.606,

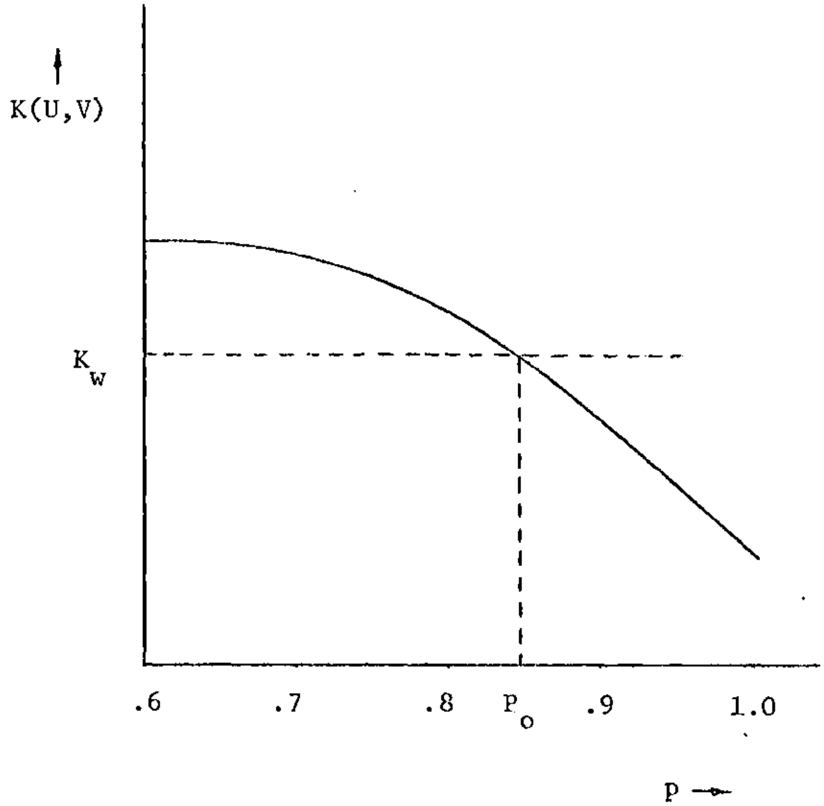

Figure 5. Cost Curve for Example 2.7

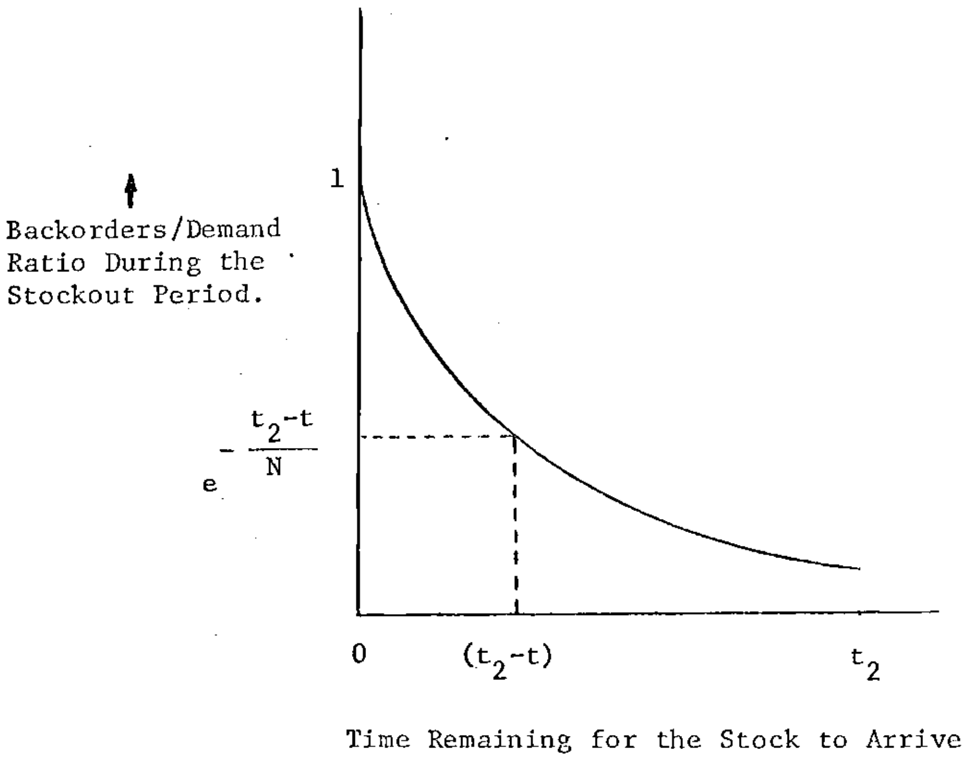

Figure 6. Relation of Backorders/Demand Ratio During the Stockout Period to the Time Remaining for the Arrival of Stock.

when $t_2 - t = \frac{10}{12}$, the fraction of demand backordered is 0.368, and so on.

The ordering and inventory carrying costs for this model will remain the same as for the other models developed earlier in this chapter. Notice that

$$\text{Fixed Shortage cost per year} = \frac{\pi S}{T}$$

$$= \frac{\pi SD}{Q+S(1-b)} \quad .$$

The backorders up to any time t are given by

$$\int_o^t De^{-\frac{t_2-t}{N}} dt \quad ,$$

or
$$= DNe^{-\frac{t_2}{N}} \left(e^{\frac{t}{N}} - 1 \right) \quad .$$

Hence the time dependent backorders cost per cycle is

$$= \bar{\pi} \int_o^{t_2} DNe^{-\frac{t_2}{N}} \left(e^{\frac{t}{N}} - 1 \right) dt$$

$$= \bar{\pi} \; DNe^{-\frac{t_2}{N}} \left(Ne^{\frac{t_2}{N}} - N - t_2 \right)$$

$$= \bar{\pi}DN \left(N - N \, e^{-\frac{t_2}{N}} - t_2 \, e^{-\frac{t_2}{N}} \right) \quad .$$

Therefore, the annual time dependent backorders cost is given by

$$\frac{\bar{\pi}DN}{T}\left(N - N e^{-\frac{t_2}{N}} - t_2 e^{-\frac{t_2}{N}} \right)$$

or $$= \frac{\bar{\pi}D^2N^2}{Q+S(1-b)} - \frac{e^{-\frac{S}{ND}}}{Q+S(1-b)} \left(\bar{\pi}D^2N^2 + \bar{\pi}DNS \right) .$$

Total backorders during the stockout period are given by

$$q = \int_o^{t_2} D e^{-\frac{t_2-t}{N}} dt$$

$$= DN \left(1 - e^{-\frac{t_2}{N}} \right) .$$

And $S = Dt_2$.

Hence the lost profit per cycle is given by

$$\pi_o(S-q) = \pi_o \left[Dt_2 - DN(1 - e^{-\frac{t_2}{N}}) \right] ,$$

and the annual lost profit is

$$\frac{\pi_o}{T} \left[Dt_2 - DN(1 - e^{-\frac{t_2}{N}}) \right]$$

$$= \frac{\pi_o DS}{Q+S(1-b)} - \frac{\pi_o D^2 N}{Q+S(1-b)} + \frac{\pi_o D^2 N\, e^{-\frac{S}{ND}}}{Q+S(1-b)} \quad .$$

$$\text{Now} \quad b = \frac{q}{S} = \frac{DN\left(1 - e^{-\frac{t_2}{N}}\right)}{Dt_2}$$

$$\text{or} \qquad b = \frac{DN\left(1 - e^{-\frac{S}{ND}}\right)}{S} \tag{25}$$

The expression for average annual cost thus becomes

$$K(Q,S) = \frac{AD}{Q+S(1-b)} + \frac{IC(Q-bS)^2}{2[Q+S(1-b)]} + \frac{\pi SD}{Q+S(1-b)} + \frac{\bar{\pi} D^2 N^2}{Q+S(1-b)}$$

$$+ \frac{e^{-\frac{S}{ND}}(\bar{\pi} D^2 N^2 + \bar{\pi} DNS)}{Q+S(1-b)} + \frac{\pi_o DS}{Q+S(1-b)} - \frac{\pi_o D^2 N}{Q+S(1-b)}$$

$$+ \frac{\pi_o D^2 N\, e^{-\frac{S}{ND}}}{Q+S(1-b)} \quad .$$

$$= \frac{AD}{Q+S(1-b)} + \frac{IC(Q-bS)^2}{2[Q+S(1-b)]} + \frac{D(\pi + \pi_o)S}{Q+S(1-b)}$$

$$+ \frac{e^{-\frac{S}{ND}}}{Q+S(1-b)} \quad (\pi_o D^2 N - N^2 D^2 \bar{\pi} - \bar{\pi} DNS)$$

$$+ \frac{\bar{\pi} D^2 N^2}{Q+S(1-b)} - \frac{\pi_o D^2 N}{Q+S(1-b)} \quad .$$

Again putting

$$U = Q+S(1-b) \quad ,$$

and $$V = Q - Sb \quad ,$$

the transformed cost equation becomes

$$K(U,V) = \frac{AD}{U} + \frac{ICV^2}{2U} + \frac{D(\pi + \pi_o)(U-V)}{U}$$

$$+ \frac{e^{-\frac{U-V}{ND}}}{U} \quad [\pi_o D^2 N - N^2 D^2 \bar{\pi} - \bar{\pi} DN(U-V)]$$

$$+ \frac{\bar{\pi} D^2 N^2}{U} - \frac{\pi_o D^2 N}{U} \quad . \tag{26}$$

A necessary condition that U* and V* be optimal is that they satisfy

$$\frac{\partial K}{\partial U} = 0 \quad ,$$

and $$\frac{\partial K}{\partial V} = 0 \quad .$$

$$\frac{\partial K}{\partial U} = \frac{1}{U^2} \Bigg[U \Bigg[D(\pi + \pi_o) + e^{-\frac{U-V}{ND}} (-\pi DN) + e^{-\frac{U-V}{ND}} \left(-\frac{1}{ND}\right) [\pi_o D^2 N$$

$$- N^2 D^2 \bar{\pi} - \bar{\pi} DN(U-V)] \Bigg] - \Bigg[AD + \frac{ICV^2}{2} + D(\pi + \pi_o)(U-V)$$

$$+ e^{-\frac{U-V}{ND}} [\pi_o D^2 N - N^2 D^2 \bar{\pi} - \bar{\pi} DN(U-V)] + \bar{\pi} D^2 N^2 - \pi_o D^2 N \Bigg] \Bigg] = 0$$

$$= \frac{1}{U^2}\left[e^{-\frac{U-V}{ND}} [U\bar{\pi}ND + U\bar{\pi}(U-V) - U\pi_o D - \pi_o D^2 N + N^2 D^2 \bar{\pi} - \bar{\pi}DNV] \right.$$

$$- AD - \frac{ICV^2}{2} + D(\pi + \pi_o)\ V - \pi D^2 N$$

$$\left. + \pi_o D^2 N \right] = 0\ , \tag{27}$$

and
$$\frac{\partial K}{\partial V} = \frac{1}{U}\left[ICV - (\pi + \pi_o)D + e^{-\frac{U-V}{ND}} (\bar{\pi}DN) \right.$$

$$\left. + e^{-\frac{U-V}{ND}} \left(\frac{1}{ND}\right)[\pi_o D^2 N - N^2 D^2 \bar{\pi} - \bar{\pi}DN(U-V)] \right] = 0\ ,$$

$$= \frac{1}{U}\left[ICV - (\pi + \pi_o)D + e^{-\frac{U-V}{ND}} [\pi_o D - \bar{\pi}(U-V)] \right] = 0 \tag{28}$$

It is evident that solving (27) and (28) analytically, in order to obtain the optimal values of U and V in terms of other system parameters is not straightforward. It might be possible to obtain an analytical solution by approximating the exponential term by a quadratic or by expanding it in a power series and taking only the first few terms into consideration. However, in this study an iterative procedure will be used to find the minimum of (26). There are several efficient search procedures developed recently to find the minimum or maximum of a function of several variables. Among these search procedures are Powell's method (12), Fletcher and Powell's method (6), Hooke and Jeeves' pattern search (9), and the sequential simplex pattern search (1), (2), (11) and

(13). Of these, the sequential simplex pattern search method is probably the simplest. Therefore, a computer search technique based on the sequential simplex will be used to find the optimal values of U and V which minimize the cost given by (26). A FORTRAN program has been prepared for this procedure and is listed at Appendix I, along with the necessary instructions to prepare the card deck. It should be noted that the solution obtained by this search procedure may not always yield an absolute minimum. However, an absolute minimum perhaps can be found by taking several randomly chosen starting points.

Example 2.8 Consider an item with the following characteristics:

$D = 200$ $C = \$25$

$I = 0.20$ $A = \$5$

$\pi = \$0.20$ $\bar{\pi} = \$10$

$\pi_o = \$12$

The computer program for this problem was run using different values of N, ranging from 0.01 to 7.0 . The computer output giving the final optimal results is shown in Appendix II and the relationship between N and the optimal K(U,V) is plotted in Figure 7. From this it can be seen that the optimal values of K(U,V), as obtained from this search procedure, will never exceed the value of K_w. Moreover, the higher the value of N, the lower the system cost K(U,V) will be.

Example 2.9 Consider an inventory system with the following parameters:

$D = 350$ $A = \$5$

$C = \$30$ $I = 0.20$

$\pi = \$0.25$ $\bar{\pi} = \$7$

$\pi_o = \$10$

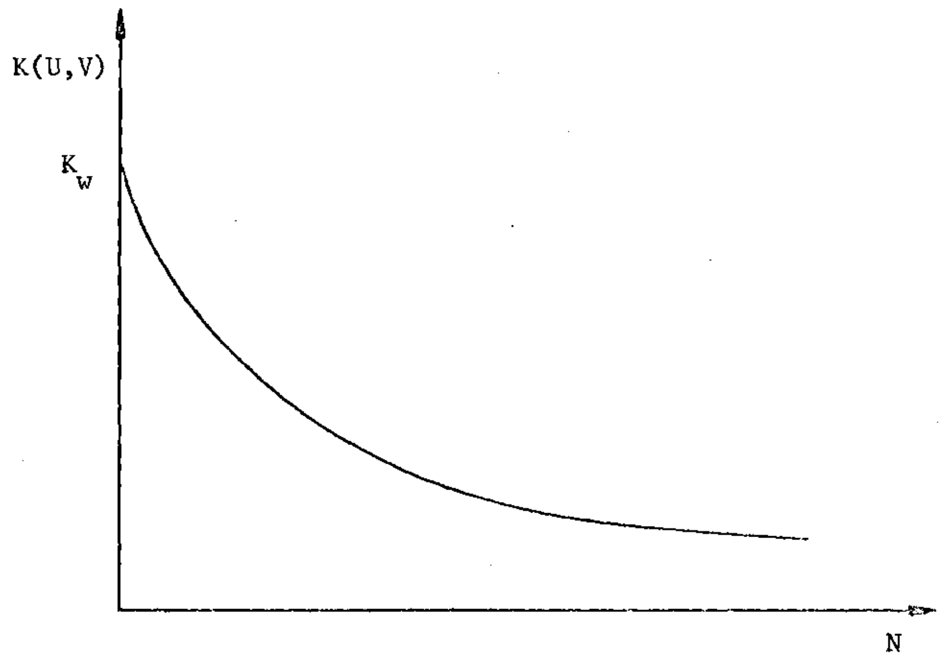

Figure 7. Cost Curve for Example 2.8

It is also known that during the stockout period, only 50% of the customers prefer to wait if the arrival of stock is two weeks away and this percentage increases exponentially as the time for arrival of stock approaches.

Now, $0.5 = e^{-\frac{2}{52N}}$

hence $N = 0.555$

For this value of N, the optimal results as obtained from the computer output are

$$Q^* = 24.85 \text{ , } S^* = 0.25$$

and $K^* = 144.6$

In order to see if these results yield an absolute minimum, one may obtain the second partial derivatives as follows:

$$\frac{\partial^2 K}{\partial U^2} = \frac{1}{U} \left[e^{-\frac{U-V}{ND}} \left(\bar{\pi} + \frac{\pi_o}{N} - \frac{\bar{\pi}(U-V)}{ND} \right) \right] ,$$

$$\frac{\partial^2 K}{\partial V^2} = \frac{1}{U} \left[IC + e^{-\frac{U-V}{ND}} \left(\bar{\pi} + \frac{\pi_o}{N} - \frac{\bar{\pi}(U-V)}{ND} \right) \right] ,$$

and

$$\frac{\partial^2 K}{\partial U \, \partial V} = \frac{1}{U} \left[- e^{-\frac{U-V}{ND}} \left(\bar{\pi} + \frac{\pi_o}{N} - \frac{\bar{\pi}(U-V)}{ND} \right) \right] .$$

From the above it is evident that if

$$\bar{\pi} + \frac{\pi_o}{N} - \frac{\bar{\pi}(U-V)}{ND} \geq 0 ,$$

(26) will always be a convex function. In this example

$$\bar{\pi} + \frac{\pi_o}{N} - \frac{\bar{\pi}(U-V)}{ND} = 7 + \frac{10}{.555} - \frac{7(.25)}{.555(350)} > 0 ,$$

therefore K(U,V) will be strictly convex and the optimal results will yield an absolute minimum.

CHAPTER III

STOCHASTIC SINGLE ITEM MODELS

3.1 Introduction

In the previous chapter this research was concerned with the single item inventory system where the demand rate and procurement lead time were deterministic and constant. This chapter, however, will deal with the analysis of those inventory systems where the demand per unit time and procurement lead time are independent random variables having known probability distributions. The objective will be to determine the optimal operating doctrine which will minimize the total expected cost or maximize the total expected profit.

Two types of models will be formulated in this chapter. The first is a Lot Size Reorder Point Model, or the (Q,r) Model, in which a quantity Q is ordered each time the inventory level reaches the reorder point r. The second type is a Periodic Review Model, or the (R,T) Model, where at each review time a sufficient quantity is ordered to bring the inventory position, or the amount on hand plus on order, up to a level R. It will be assumed that the fraction of demand backordered during the stockout period is known and remains a constant throughout the stockout period. The approximate stochastic models presented by Hadley and Whitin (7) for the cases when $b = 1$ (all the demands during stockout period are backordered) and $b = 0$ (all the demands during stockout period are lost) will be drawn upon and modified in order to develop models for the case $0 \leq b \leq 1$.

3.2 The Lot Size - Reorder Point (Q,r) Model

As mentioned earlier, in this type of model a quantity Q is ordered each time the inventory level reaches the reorder point r. Such an operating policy implies continuous review of the system so that there is no overshoot of the reorder point, and is generally called "transactions reporting." In this section a single item model will be developed for the inventory system using transactions reporting. It should, however, be noted that this development will not be based upon the exact formulation but involves a heuristic approximate treatment and requires that the following assumptions be made:

1. The demand is a continuous random variable.
2. The unit cost of the item is a constant independent of the order quantity Q.
3. There is a fixed shortage cost π for each unit of demand occurring during the stockout period whether that unit is backordered or lost.
4. The number of units demanded per demand is small so that there is no overshoot of reorder point.
5. There is no time dependent backorder cost, i.e. $\hat{\pi} = 0$.
6. The reorder point r, based on the net inventory is positive.
7. There is never more than a single order outstanding.
8. The stockout period during a cycle is small enough to be neglected so that the average number of cycles per year is $\frac{D}{Q}$, where D is the average annual demand.

The objective is to determine the optimal order quantity Q and the reorder point r for a given item. In the previous chapter it was seen that with the proper definition of stockout cost, the minimization of the average annual cost is equivalent to maximization of the average annual profit. The optimal values of Q and r, therefore, will be found by minimizing the average annual cost model, the formulation of which will closely follow Hadley and Whitin's heuristic approximate treatment of (Q,r) models (7).

By definition let $f(x;t)dx$ be the probability that the number of units demanded in a time t lies between x and x + dx. Now, if the procurement lead time is a random variable such that $g(\bar{t})d\bar{t}$ is the probability that the procurement lead time lies between $\bar{t}$ and $\bar{t} + d\bar{t}$, then the marginal distribution of lead time demand is given by

$$h(x) = \int_0^\infty f(x;\bar{t})g(\bar{t})d\bar{t} \tag{31}$$

If, however, the procurement lead time is a constant, say $\bar{t}$, the marginal distribution of lead time demand becomes

$$h(x) = f(x;\bar{t}) \ . \tag{32}$$

If A is the cost of placing an order, then since the average annual demand is D and since an order is placed after every Q demands, the average annual cost of placing orders is $\frac{DA}{Q}$. If the lead time demand is x then the expected demand short at the end of the cycle is given by

$$\bar{n}(r) = \int_r^\infty (x-r)h(x)dx \ .$$

Hence the expected number of backorders per cycle are

$$b \int_r^{\infty} (x-r)h(x)dx ,$$

where b is the ratio of the expected number of backorders to the expected number of demand short per cycle. Therefore, one can see that the expected demand lost per cycle is

$$(1-b) \int_r^{\infty} (x-r)h(x)dx .$$

The expected net inventory at the beginning of the cycle, assuming that the arrival of an order initiates a cycle is given by

$$Q + r-\mu + (1-b) \int_r^{\infty} (x-r)h(x)dx ,$$

where μ is the expected lead time demand. Also the expected net inventory at the end of the cycle is given by

$$r - \mu + (1-b) \int_r^{\infty} (x-r)h(x)dx .$$

These will also be the expected values of the on hand inventory at the above times if one neglects the expected number of backorders. Since the mean rate of demand remains constant, the expected on hand inventory will decrease linearly from $Q + r - \mu + (1-b) \int_r^{\infty} (x-r)h(x)dx$ at the beginning of the cycle to $r - \mu + (1-b) \int_r^{\infty} (x-r)h(x)dx$ at the end of the cycle and will average to

$$\frac{Q}{2} + r - \mu + (1-b) \int_r^{\infty} (x-r)h(x)dx .$$

The annual cost of carrying inventory, therefore, is given by

$$IC \left(\frac{Q}{2} + r - \mu \right) + IC(1-b) \int_r^{\infty} (x-r)h(x)dx .$$

Since the expected demand short at the end of the cycle is

$$\bar{n}(r) = \int_r^\infty (x-r)h(x)\,dx \ ,$$

the fixed shortage cost per cycle becomes

$$\pi \int_r^\infty (x-r)h(x)\,dx \ ,$$

and therefore, the annual fixed shortage cost is given by

$$\frac{\pi D}{Q} \int_r^\infty (x-r)h(x)\,dx \quad .$$

Also, since the expected demand lost per cycle is

$$(1-b)\int_r^\infty (x-r)h(x)\,dx \quad ,$$

the annual lost profit is given by

$$\frac{\pi_o D(1-b)}{Q} \int_r^\infty (x-r)h(x)\,dx \quad .$$

Therefore the annual stockout cost becomes

$$\frac{\pi D}{Q} \int_r^\infty (x-r)h(x)\,dx + \frac{\pi_o D(1-b)}{Q} \int_r^\infty (x-r)h(x)\,dx \quad ,$$

or $$[\pi + \pi_o(1-b)]\frac{D}{Q} \int_r^\infty (x-r)h(x)\,dx \quad .$$

All the components of the average annual variable cost $K(Q,r)$ have been found. $K(Q,r)$ is just the sum of the above components, or

$$K(Q,r) = \frac{AD}{Q} + IC\left(\frac{Q}{2} + r - \mu\right)$$

$$+ \left(IC(1-b) + \frac{\pi D}{Q} + \frac{\pi_o(1-b)D}{Q} \right) \int_r^\infty (x-r)h(x)\,dx \qquad (33)$$

The objective is to determine the optimal values of Q and r, say Q^* and r^*, which minimize $K(Q,r)$. If the optimal Q^* and r^* satisfy $0<Q^*<\infty$, and $0<r^*<\infty$ respectively, then Q^* and r^* must satisfy the following

$$\frac{\partial K}{\partial Q} = 0 ,$$

and

$$\frac{\partial K}{\partial r} = 0 .$$

These derivatives are

$$\frac{\partial K}{\partial Q} = 0 = -\frac{AD}{Q^2} + \frac{IC}{2} - \left(\frac{\pi D + \pi_o(1-b)D}{Q^2}\right) \bar{n}(r) , \tag{34}$$

and

$$\frac{\partial K}{\partial r} = 0 = IC + \left(IC(1-b) + \frac{\pi D}{Q} + \frac{\pi_o(1-b)D}{Q}\right) \left(-rh(r) + rh(r) - H(r)\right) \tag{35}$$

where $H(x)$ is the complementary cumulative of $h(x)$, i.e.

$H(x) = \int_x^\infty h(x)dx$. From (34) and (35) one may obtain

$$Q = \sqrt{\frac{2D[A + \pi\bar{n}(r) + \pi_o(1-b)\bar{n}(r)]}{IC}} , \tag{36}$$

and $$H(r) = \frac{QIC}{QIC(1-b) + \pi D + \pi_o(1-b)D} \tag{37}$$

It will be noted that (37) does not make any sense if

$$\frac{QIC}{QIC(1-b) + \pi D + \pi_o(1-b)D} > 1 .$$

In such a case no solution will exist. This situation is easily explained, because

$$\frac{QIC}{QIC(1-b) + \pi D + \pi_o(1-b)D} > 1$$

means that the inventory carrying cost is greater than the shortage cost and therefore, the item should not be stocked at all. One may show that the solution Q* and r* obtained from (36) and (37) gives the absolute minimum of the cost function K(Q,r). The proof of this follows from the fact that K(Q,r) is convex.

The cost function K(Q,r) given by (33) can be considered as the sum of two function K_1 and K_2 such that

$$K_1 = \frac{AD}{Q} + IC\left(\frac{Q}{2} + r - \mu\right) ,$$

and

$$K_2 = \left[IC(1-b) + \frac{\pi D}{Q} + \frac{\pi_o(1-b)D}{Q}\right] \int_r^\infty (x-r)h(x)\,dx.$$

A necessary and sufficient condition for a function K(Q,r) to be convex is that

$$\frac{\partial^2 K}{\partial Q^2} \geq 0 , \tag{38}$$

$$\frac{\partial^2 K}{\partial r^2} \geq 0 , \tag{39}$$

and

$$\frac{\partial^2 K}{\partial Q^2}\,\frac{\partial^2 K}{\partial r^2} - \left(\frac{\partial^2 K}{\partial Q \partial r}\right)^2 \geq 0 . \tag{40}$$

In the case of K_1 one has

$$\frac{\partial K_1}{\partial Q} = -\frac{AD}{Q^2} + \frac{IC}{2} \quad ,$$

$$\frac{\partial^2 K_1}{\partial Q^2} = \frac{2AD}{Q^3} \quad ,$$

$$\frac{\partial K_1}{\partial r} = IC \quad ,$$

$$\frac{\partial^2 K_1}{\partial r^2} = 0 \quad ,$$

$$\frac{\partial^2 K_1}{\partial Q \partial r} = 0 \quad ,$$

and

$$\frac{\partial^2 K_1}{\partial Q^2} \frac{\partial^2 K_1}{\partial r^2} - \left(\frac{\partial^2 K_1}{\partial Q \partial r} \right)^2 = 0 \quad .$$

From the above one can observe that for $Q>0$, K_1 satisfies the conditions given by (38), (39) and (40) and therefore, it is a convex function. Similarly for K_2,

$$\frac{\partial K_2}{\partial Q} = \frac{-[\pi D + \pi_o(1-b)D]}{Q^2} \bar{n}(r) \quad ,$$

$$\frac{\partial^2 K_2}{\partial Q^2} = \frac{2[\pi D + \pi_o(1-b)D]}{Q^3} \bar{n}(r) \quad ,$$

$$\frac{\partial K_2}{\partial r} = - \left[IC(1-b) + \frac{\pi D}{Q} + \frac{\pi_o (1-b) D}{Q} \right] H(r)$$

$$\frac{\partial^2 K_2}{\partial r^2} = \left[IC(1-b) + \frac{\pi D}{Q} + \frac{\pi_o (1-b) D}{Q} \right] h(r)$$

$$\frac{\partial^2 K_2}{\partial Q \partial r} = \frac{[\pi D + \pi_o (1-b) D]}{Q^2} H(r) \quad ,$$

and $$\frac{\partial^2 K_2}{\partial Q^2} \frac{\partial^2 K_2}{\partial r^2} - \left(\frac{\partial^2 K_2}{\partial Q \partial r} \right)^2 =$$

$$\frac{2[\pi D + \pi_o (1-b) D]}{Q^3} \bar{n}(r) \left(IC(1-b) + \frac{\pi D}{Q} + \frac{\pi_o (1-b) D}{Q} \right) h(r)$$

$$- \left(\frac{\pi D + \pi_o (1-b) D}{Q^2} \right)^2 [H(r)]^2 \quad .$$

From the above it can be shown that K_2 satisfies the conditions (38), (39) and (40) for all values of $Q>0$ and $r>0$, and hence it is a convex function. Now, since the sum of any two convex functions is also a convex function (8), it can be concluded that $K(Q,r)$ is a convex function.

Finally one may note that if $h(x)$ is assumed to be a normal distribution with mean μ(the expected lead time demand) and standard deviation σ, then

$$K(Q,r) = \frac{AD}{Q} + IC \left(\frac{Q}{2} + r - \mu\right) + \left(IC(1-b) + \frac{\pi D}{Q} + \frac{\pi_o (1-b) D}{Q} \right) \left((\mu - r)\, \Phi \left(\frac{r-\mu}{\sigma} \right) + \sigma \phi \left(\frac{r-\mu}{\sigma} \right) \right) , \qquad (41)$$

where $\phi(x)$ is the density function of standard normal distribution and $\Phi(x)$ the complementary cumulative of $\phi(x)$.

3.3 A Procedure for Solving Equations (36) and (37)

The procedure for solving the pair of equations (36) and (37) will be the same as described by Hadley and Whitin (7). Use Q_w (optimal lot size given by the Wilson formula) as the initial estimate of Q, i.e. write $Q_1 = Q_w$. Then use Q_1 in (37) to compute r_1. The r_1 so obtained is used in (36) to compute Q_2. This Q_2 is used in (37) to compute r_2, etc.. This iterative procedure is continued until Q and r are obtained with sufficient accuracy.

For the curve described by equation (37) in the Qr plane, one can have, when $Q = 0$, $r = \infty$ and when

$$Q = \frac{\pi D + \pi_o(1-b)D}{ICb} \quad , \quad r = 0 \ .$$

Furthermore, to ascertain the slope of the curve,

$$\frac{dQ}{dr} = \frac{-h(r)[\pi D + \pi_o(1-b)D]\ [1-(1-b)H(r) + (1-b)H(r)]}{IC[1-(1-b)H(r)]^2} \ ,$$

or

$$\frac{dr}{dQ} = -\frac{IC[1-(1-b)H(r)]^2}{[\pi D + \pi_o(1-b)D]h(r)} < 0 \ .$$

Similarly for the curve described by (36), when $r = \infty$, $Q = Q_w$ and when $r = 0$,

$$Q = \bar{Q} = \sqrt{\frac{2D\left(A + \mu\ [\pi + (1-b)\pi_o]\right)}{IC}} \ .$$

Furthermore,

$$\frac{dQ}{dr} = \frac{1}{2}\sqrt{\frac{IC}{2D[A + \pi\bar{n}(r) + \pi_o(1-b)\bar{n}(r)]}}\left[-H(r)\frac{2D}{IC}[\pi + \pi_o(1-b)]\right] ,$$

$$= \sqrt{\frac{D}{2IC[A + \pi\bar{n}(r) + \pi_o(1-b)\bar{n}(r)]}} \left[\left(\pi + \pi_o(1-b)\right) H(r) \right] ,$$

or,

$$\frac{dr}{dQ} = -\sqrt{\frac{2IC[A + \pi\bar{n}(r) + \pi_o(1-b)\bar{n}(r)]}{D}} \left[\left(\frac{1}{\pi + \pi_o(1-b)} \right) \frac{1}{H(r)} \right] ,$$

$$< 0 \quad .$$

The graphical representation of the two curves is shown at Figure 8. It can be seen that the two curves will intersect only if

$$\bar{Q} < \frac{\pi D + \pi_o(1-b)D}{ICb} \quad .$$

Should $\bar{Q} > \dfrac{\pi D + \pi_o(1-b)D}{ICb}$, it will mean that equations (36) and (37) do not have a solution.

Example 3.1 Consider an inventory system with the following parameters:

$D = 1600$	$C = \$50$
$A = \$2500$	$I = 1.00$
$\pi = \$100$	$\pi_o = \$50$
$b = 0.5$	

Also, the marginal distribution of lead time demand is normally distributed with mean 300 units and standard deviation 25 units.

$$Q_1 = Q_W = \sqrt{\frac{2AD}{IC}} ,$$

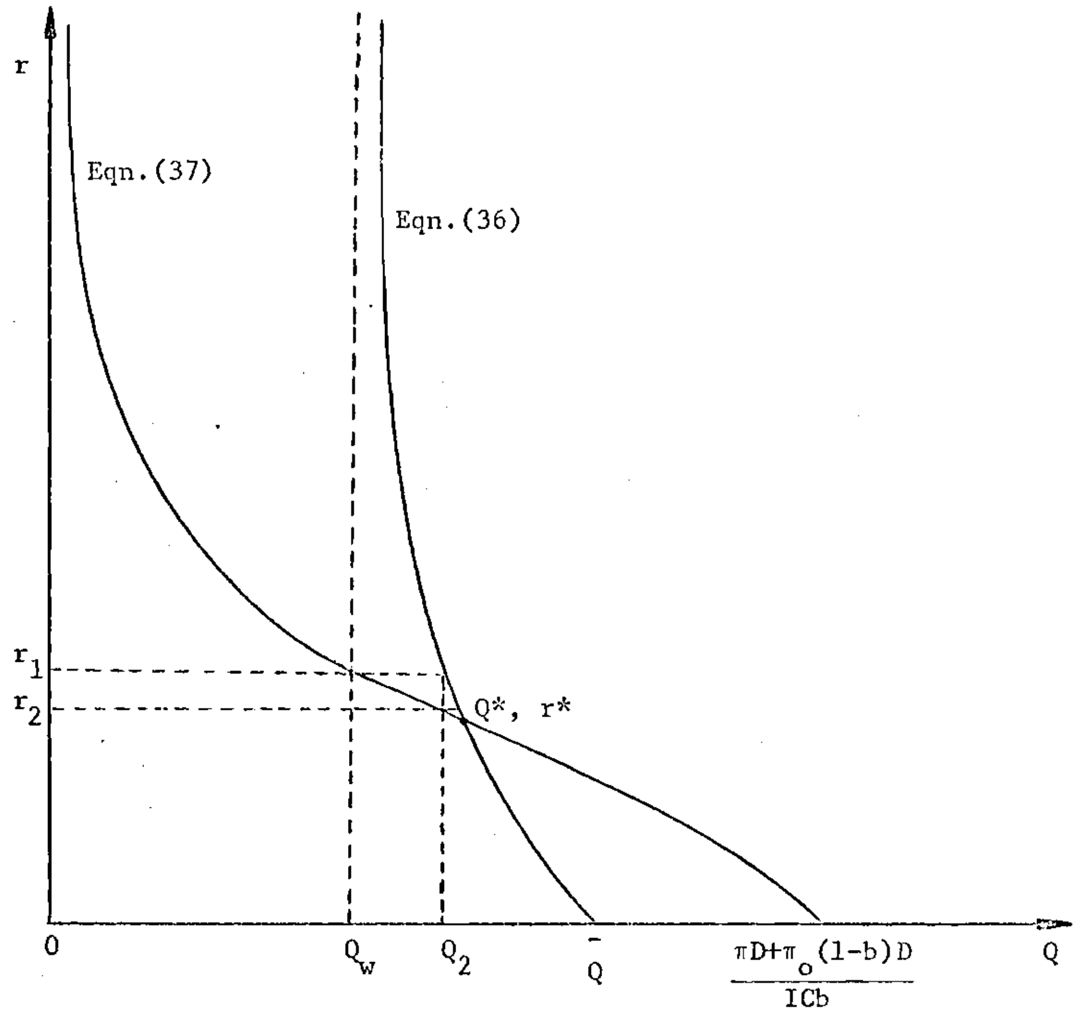

Figure 8. Behavior of Equations (36) and (37).

$$= \sqrt{\frac{2(2500)(1600)}{(1.0)(50)}} = 400 .$$

From (37) one can obtain

$$H(r_1) = \Phi\left(\frac{(r_1 - 300)}{25}\right) = \frac{Q_1 IC}{Q_1 IC(1-b) + \pi D + \pi_o(1-b)D} ,$$

$$= \frac{400(50)}{400(50)(.5) + (100)(1600) + 50(.5)(1600)}$$

$$= 0.0953 ,$$

hence from the normal tables

$$\frac{r_1 - 300}{25} = 1.31 ,$$

or $r_1 = 332.8$.

Now $\bar{n}(r_1) = (\mu - r_1) \quad \Phi\left(\frac{r_1 - \mu}{\sigma}\right) + \sigma\,\phi\left(\frac{r_1 - \mu}{\sigma}\right)$

$$= (-32.8)(.0953) + 25(.1691)$$

$$= 1.10 .$$

From (36) ,

$$Q_2 = \sqrt{\frac{2D[A + \pi\bar{n}(r_1) + \pi_o(1-b)\bar{n}(r_1)]}{IC}}$$

$$= \sqrt{\frac{2(1600)[2500 + 100(1.10) + 50(.5)(1.10)]}{50}}$$

$$= 410.5$$

Using this value of Q_2 in (37) gives

$$\Phi\left(\frac{r_2 - 300}{25}\right) = 0.0976$$

or from the normal tables $\dfrac{r_2 - 300}{25} = 1.295$,

and, therefore, $r_2 = 332.4$

No additional iterations will be needed since the changes in Q and r are negligible. The optimal values thus, are

$$Q^* = 410.5 \ ,$$

and $$r^* = 332.8 \ .$$

The average annual cost is computed from (41) and is given by

$$K(Q,r) = \frac{2500(1600)}{410.5} + 50\left(\frac{410.5}{2} + 332.8 - 300\right)$$

$$+ \left(50(.5) + \frac{100(1600)}{410.5} + \frac{50(.5)(1600)}{410.5}\right)$$

$$\left[(-32.4)(.0953) + 25(.1691)\right] \ ,$$

$$= 22203 \ .$$

In order to observe the effect of b on the system cost $K(Q,r)$, the above problem was solved for the cases when $b = 0$ and when $b = 1$. The following results were obtained:

when $b = 0$, $K(Q,r) = 22319$,

and when $b = 1$, $K(Q,r) = 22031$.

One can see from the results that the cost $K(Q,r)$ is minimum when $b = 1$. The cost increases as the value of b decreases, and reaches a maximum when $b = 0$. This is obvious because with $\bar{\pi} = 0$, the stockout cost per unit of demand is equal to $\pi + \pi_0(1-b)$. It will be minimum when $b = 1$ or maximum when $b = 0$.

3.4 The Periodic Review (R,T) Model

In periodic review inventory systems, the system is reviewed after every time interval of length T, and at each review time a sufficient quantity is ordered to bring the inventory position up to a level R. The essential difference between the periodic review system and the transaction reporting system arises from the fact that whereas transaction reporting requires the system to be reviewed after every transaction, in periodic review system, as the name implies, the system is reviewed periodically.

In order to find the optimal values of R and T, this section will present the formulation and analysis of a simple, approximate periodic review model which will closely follow Hadley and Whitin's treatment of simple, approximate (R,T) models (7). The formulation will be based on the following assumptions:

1. The demand is a continuous random variable.
2. The cost J of making a review is independent of the variables R and T.
3. The unit cost of the item is a constant, independent of the quantity ordered.
4. There is a fixed shortage cost π for each unit of demand occurring during the stockout period whether that unit is backordered or lost.
5. There is no time dependent backorder cost.
6. The backorders are incurred in very small quantities

so that when an order arrives, it is almost always sufficient to meet any outstanding backorders.

7. When the procurement lead time is a random variable, it is assumed that the orders are received in the same sequence in which they are placed, and furthermore, lead times for different orders can be treated as independent random variables.

Let $f(x;t)$ be the density function for the demand x in a time interval of length t. Also, let D be the average demand rate.

The annual ordering and review cost is given by $\frac{L}{T}$, where $L = A + J$. For computing the inventory carrying cost, the period T will be used as the time between the arrival of two successive orders rather than between the placement of two successive orders.

Suppose now that the procurement lead time $\bar{t}$ is a random variable with density $g(\bar{t})$ and let $\bar{t}_{min}$ and $\bar{t}_{max}$ be the lower and upper limits respectively to the possible range of lead time values. Then if $\bar{t}_1$ and $\bar{t}_2$ are the lead times for the orders placed at t and t + T respectively, the expected number of demands short at the end of each period must be

$$\int_{\bar{t}_{min}}^{\bar{t}_{max}} \int_{\bar{t}_{min}}^{\bar{t}_{max}} \int_{R}^{\infty} (x-R) f(x;\bar{t}_2+T) g(\bar{t}_2) g(\bar{t}_1) dx \, d\bar{t}_2 \, d\bar{t}_1$$

or

$$\int_{R}^{\infty} (x-R)\bar{h}(x;T)\,dx$$

where
$$\bar{h}(x;T) = \int_{\bar{t}_{min}}^{\bar{t}_{max}} f(x;\bar{t}_2+T) g(\bar{t}_2) d\bar{t}_2 \qquad (42)$$

This follows since $\int_{\bar{t}_{min}}^{\bar{t}_{max}} g(\bar{t}_1) d\bar{t}_1 = 1$.

If by definition $\bar{h}(x;T)$, is $f(x;\bar{t}+T)$ when the procurement lead time is a constant and is (42) when procurement lead time is a random variable then in both cases, the expected number of demands short per period is given by

$$\int_{R}^{\infty} (x-R)\bar{h}(x;T)\,dx \quad .$$

The expected number of demands backordered per period is

$$b \int_{R}^{\infty} (x-R)\bar{h}(x;T)\,dx \ ,$$

and the expected number of demands lost per period is

$$(1-b) \int_{R}^{\infty} (x-R)\bar{h}(x;T)\,dx \ .$$

Therefore, the expected net inventory at the beginning of the period is

$$R - \mu + (1-b) \int_{R}^{\infty} (x-R)\bar{h}(x;T)\,dx \ , \tag{43}$$

and the expected net inventory at the end of the period is

$$R - \mu - DT + (1-b) \int_{R}^{\infty} (x-R)\bar{h}(x;T)\,dx \quad . \tag{44}$$

Since the mean rate of demand is constant, the expected net inventory will decrease from (43) at the beginning of the period to (44) at the end of the period and will average to

$$R - \mu - \frac{DT}{2} + (1-b) \int_{R}^{\infty} (x-R)\bar{h}(x;T)\,dx \quad .$$

It is assumed that the backorders are incurred only in very small quantities and therefore, the integral over time of the net inventory must very closely approximate the integral over time of the on hand inventory. The average annual cost of carrying inventory, therefore, becomes

$$IC \left[R - \mu - \frac{DT}{2} + (1-b) \int_R^\infty (x-R)\bar{h}(x;T)\,dx \right] .$$

The annual expected fixed shortage cost is given by

$$\frac{\pi}{T} \int_R^\infty (x-R)\bar{h}(x;T)\,dx,$$

and the annual expected lost profit is

$$\frac{\pi_o(1-b)}{T} \int_R^\infty (x-R)\bar{h}(x;T)\,dx .$$

Therefore, the annual expected stockout cost is given by

$$\frac{\pi + \pi_o(1-b)}{T} \int_R^\infty (x-R)\bar{h}(x;T)\,dx .$$

The average annual variable cost can now be written as the sum of the above components, or

$$K(R,T) = \frac{L}{T} + IC \left(R - \mu - \frac{DT}{2} \right) + \left[IC(1-b) + \frac{\pi + \pi_o(1-b)}{T} \right] \int_R^\infty (x-R)\bar{h}(x;T)\,dx . \tag{45}$$

For a given T, the value of R which minimizes (45) must satisfy

$$\frac{\partial K}{\partial R} = 0 = IC - \left[IC(1-b) + \frac{\pi + \pi_o(1-b)}{T} \right] \bar{H}(R;T) ,$$

where $\bar{H}(R;T) = \int_R^\infty \bar{h}(x;T)\,dx$ is the complementary cumulative of $\bar{h}(x;T)$.

Thus R^* , the optimal value of R is a solution to

$$\bar{H}(R;T) = \frac{ICT}{ICT(1-b) + \pi + \pi_o(1-b)} \quad . \tag{46}$$

It is evident from (46) that no solution will exist if

$\dfrac{ICT}{ICT(1-b) + \pi + \pi_o(1-b)} > 1$, because in such a case the inventory carrying cost will be greater than the shortage cost and therefore the item should not be stocked at all.

One may show that for a given T, $0<T<\infty$, the solution obtained from (46), i.e. R^* will yield an absolute minimum. This follows from the fact that for a given T, K(R,T) is a convex function of R because

$$\frac{\partial^2 K}{\partial R^2} = \left[IC(1-b) + \frac{\pi + \pi_o(1-b)}{T} \right] \bar{h}(R;T) > 0 \quad .$$

The procedure for determining the optimal value of T will be the same as followed by Hadley and Whitin (7), i.e. to tabulate K(R,T) as a function of T, using the R^* for given T in computing K(R;T), plot the results and in this way determine T^*.

Example 3.2 Consider an inventory system with the following parameters:

$I = 0.30$		$C = \$50$	
$T = 6$ months		$\bar{t} = 3$ months	
$\pi = \$50$		$\pi_o = \$30$	
$b = 0.5$		$L = \$25$	

The demand during any time interval of length t is normally distributed with mean 200t and variance 400t. It is desired to determine the optimal value of R.

Expected demand in time $T + \bar{t} = 200(3/4) = 150$, and variance of demand in this time $= 400(3/4) = 300$, or the standard deviation $= \sqrt{300} = 17.3$.

From (46)

$$\Phi\left(\frac{R - 150}{17.3}\right) = \frac{(.30)(50)(.5)}{(.30)(50)(.5)(.5) + 50 + 30(.5)}$$

$$= 0.109 ,$$

and hence from the normal tables,

$$\frac{R - 150}{17.3} = 1.23 ,$$

or

$$R = 171.3 .$$

For the normal distribution

$$\int_R^{\infty} (x-R)\bar{h}(x;T)\,dx = (\mu - R)\ \Phi\left(\frac{R - \mu}{\sigma}\right) + \sigma\phi\left(\frac{R - \mu}{\sigma}\right)$$

$$= (-21.3)(.109) + 17.3(.1872)$$

$$= 0.92 .$$

Therefore, from (45)

$$K(R,T) = \frac{25}{.5} + (.30)(50)(171.3 - 50 - \frac{200}{2})$$

$$+ \left[(.30)(50)(.5) + \frac{50 + 30(.5)}{.5}\right] (.92)$$

$$= 496 .$$

In order to observe the behavior of system cost K(R,T) with respect to b, the above problem was solved for the cases when b = 0 and b = 1, and the following results were obtained,

when b = 0 K(R,T) = 525

when b = 1 K(R,T) = 452 .

It can be seen from the results that for a given value of T, K(R,T) increases as the value of b decreases from 1 to 0. This is because of the fact that the stockout cost per unit of demand, which is equal to $\pi + \pi_o(1-b)$, will be minimum when b = 1 or maximum when b = 0.

CHAPTER IV

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

4.1 Summary

The objective of this investigation was to study the single item inventory system in which during the stockout period, a fraction of the demand is backordered and the remaining is lost. The analysis of the system includes the formulation of an average annual cost model for the system and then minimizing this cost function in order to obtain the optimal operating doctrine. It is shown that with the proper definition of stockout cost, the minimization of average annual cost will be equivalent to maximization of average annual profit.

For the deterministic system three types of models have been formulated. The first model is based on the assumption that during the stockout period the ratio of backorders to demand remains a constant. The formulation of a second deterministic model takes into account the assumption that at the point in time when the system just goes out of stock, the ratio of backorders to demand is known and is equal to p. This ratio increases linearly during the stockout period and at the end of the cycle, when the stock is just about to arrive, the ratio equals 1. For both of these models, the average annual cost is expressed in terms of the ordering cost, the inventory carrying cost, and the shortage cost and is then minimized in order to obtain the optimal solution. It is found that the optimal solution yields an absolute minimum. In the third deterministic model, the backorders to demand ratio during the stockout period depends upon the time remaining for the stock to arrive and increases exponentially as the time for the arrival of stock approaches. The average annual cost for this model is formulated but it is found that obtaining an analytical solution to this model is not straightforward. As an alternative a computer search technique based on the sequential simplex pattern search is used to find the optimal solution.

Two types of stochastic inventory models have been formulated. The first is a Lot Size Reorder Point model, or the (Q,r) model, in which a quantity Q is ordered each time the inventory level reaches the reorder level r. The second type is a Periodic Review model, or the (R,T) model, where at each review time a sufficient quantity is ordered to bring the inventory position up to a level R. It is assumed that the fraction of demand backordered during the stockout period is known and remains a constant throughout the stockout period. For both these models, it turns out that the solution obtained herein by minimizing the average annual cost, yields an absolute minimum.

4.2 Conclusions

The inventory models presented in this study are for single item inventory system where the ratio of backorders to demand during the stockout period varies between 0 and 1. The models developed in the literature for the cases where this ratio is either equal to 0 or equal to 1 are special cases of the models formulated herein. The applicability of these models is subject to the assumptions made in Chapter II and Chapter III for the deterministic and stochastic inventory systems respectively.

In the physical inventory systems situations are generally encountered where during the stockout period some demands are backordered and some are lost. Therefore, it is felt that these models are closer to reality than other models developed for similar inventory systems and hence these may be useful for practical solutions of inventory problems. For those particular inventory systems where the variability of backorders to demand ratio during the stockout period is known but is different from the cases examined in this study, it may be possible to formulate mathematical models along the lines illustrated by this study.

4.3 Recommendations

The following recommendations are made with respect to further studies:

1. The development of a model for backorders and lost sales for inventory systems where the demand rate during the cycle is not constant but varies with a known variability.

2. The development of dynamic inventory models for backorders and lost sales. This should include seasonal variations in demand by breaking down the period into component stages.

3. The development of similar models for multi-item inventory systems with constraints.

4. Treatment of the number of backorders per cycle as a random variable. For example, if S is the total demand occurring during the stockout period and b is the probability that a demand will be backordered, then the number of backorders during the cycle, say X, will have a binomial distribution so that the probability that X = q is given by

$$\binom{S}{q} b^q \quad (1-b)^{S-q} \quad .$$

Therefore, this would require the derivation of expected total cost which, in general, will not be the same as the average annual cost derived in this study.

APPENDICES

APPENDIX I

PREPARATION OF THE PROGRAM DECK

The Nomenclature used in the writing of the program is as follows:

OBJ(U , V)	=	The expression for K(U,V)
C(1)	=	D
C(2)	=	A
C(3)	=	I
C(4)	=	C
C(5)	=	π
C(6)	=	π_o
C(7)	=	$\bar{\pi}$
C(8)	=	N
BB(U,V)	=	The expression for b
ACC	=	The fraction up to which the accuracy is desired.
BO	=	b

X1(1), X1(2) and X2(1), X2(2) and X3(1), X3(2) are the three starting points required for the program.

Key-punching the Data Cards:

Free Field Format will be used to punch data in all the data cards. In the first data card the values of system parameters D, A, I, C, π, π_o, and $\bar{\pi}$, will be punched in the order mentioned. To illustrate for the example 2.8, the first card is punched as follows:

200.0,5.0,.20,25.0,.20,12.0,10.0,

Each of the remaining cards except for the last one, will be punched with one value of N each. For example if the program is to be run for N = .1, N = .2, and N = .3, the second card will be punched as

.1,

the third card will be punched as

.2,

and the fourth card will be punched as

.3,

The last data card will always appear as

0.0,

The listing of the FORTRAN program appears on the following page.

```
C      THIS PROGRAM IS USED FOR THE PURPOSE OF MINIMIZING THE AVERAGE
C      ANNUAL COST FOR AN INVENTORY SYSTEM WHERE THE RATIO OF BACKORDERS
C      TO DEMAND INCREASES EXPONENTIALLY DURING THE STOCKOUT PERIOD.
       DIMENSION X1(2),X2(2),X3(2),X4(2),X5(2),X6(2),X7(2),X0(2)
       DIMENSION C(10)
       OBJ( U  ,  V )=  (((C(3)*C(4)* V   **2)/2.)+(C(1)*(C(5)+C(6))*
      1( U - V   ))+(EXP(-( U  -  V )/(C(8)*C(1)))*(C(6)*C(1)**2*
      2C(8)-C(8)**2*C(1)**2*C(7)-C(7)*C(1)*C(8)*(  U  -  V  )))+(C(7)*
      3C(1)**2*C(8)**2)-(C(6)*C(1)**2*C(8))+(C(2)*C(1)))/ U
       BB(U,V) =(C(1)*C(8)*(1.0-EXP(-(U-V)/(C(8)*C(1)))))/(U-V)
       READ(5,3)(C(I),I=1,7)
  3    FORMAT(      )
 3000  READ(5,3) C(8)
       IF (C(8)) 2000,9006,2000
 2000  CONTINUE
       ACC=0.5
       ITER=0
       ALPHA = 1.0
       BETA = 0.5
       GAMMA = 2.0
       X1(1)=14.0
       X1(2)=12.0
       X2(1)=10.0
       X2(2)=4.0
       X3(1)=5.0
       X3(2)=9.0
       Y1 = OBJ(X1(1),X1(2))
       Y2 = OBJ(X2(1),X2(2))
       Y3 = OBJ(X3(1),X3(2))
C       INITIALISATION COMPLETE, ITERATIONS BEGIN
    1  IF (Y1.LE.Y2) GO TO 15
       AA=X1(1)
       B = X1(2)
       CC= Y1
       X1(1) = X2(1)
       X1(2) = X2(2)
       Y1 = Y2
       X2(1)=AA
       X2(2) = B
       Y2 = CC
   15  IF (Y2.LE.Y3) GO TO 25
       AA=X2(1)
       B = X2(2)
       CC= Y2
       X2(1) = X3(1)
       X2(2) = X3(2)
       Y2 = Y3
```

```
      X3(1)=AA
      X3(2) = B
      Y3 = CC
   25 X4(1) = (X1(1) + X2(1))*0.5
      X4(2) = (X1(2) + X2(2))*0.5
      X5(1) = X4(1) + ALPHA*(X4(1) - X3(1))
      X5(2) = X4(2) + ALPHA*(X4(2) - X3(2))
      Y5 = OBJ(X5(1),X5(2))
      IF (Y5.LE.Y1) GO TO 201
      IF (Y5.LE.Y2) GO TO 301
      IF (Y5.GE.Y3) GO TO 101
      Y3 = Y5
      X3(1) = X5(1)
      X3(2) = X5(2)
  101 X7(1) = X4(1) + BETA*(X3(1) - X4(1))
      X7(2) = X4(2) + BETA*(X3(2) - X4(2))
      Y7 = OBJ(X7(1),X7(2))
      IF (Y7.GE.Y3) GO TO 401
      X3(1) = X7(1)
      X3(2) = X7(2)
      Y3 = Y7
      GO TO 1000
  201 X6(1) = X4(1) + GAMMA*(X5(1) - X4(1))
      X6(2) = X4(2) + GAMMA*(X5(2) - X4(2))
      Y6 = OBJ(X6(1),X6(2))
      IF (Y6.LE.Y1) GO TO 501
  301 X3(1) = X5(1)
      X3(2) = X5(2)
      Y3 = Y5
      GO TO 1000
  401 X2(1) = (X2(1) + X1(1))/2.0
      X2(2) = (X2(2) + X1(2))/2.0
      X3(1) = (X3(1) + X1(1))/2.0
      X3(2) = (X3(2) + X1(2))/2.0
      Y2 = OBJ(X2(1),X2(2))
      Y3 = OBJ(X3(1),X3(2))
      GO TO 1000
  501 X3(1) = X6(1)
      X3(2) = X6(2)
      Y3 = Y6
      GO TO 1000
 1000 ITER = ITER + 1
      IF (ABS(Y1-Y2).GE.ACC) GO TO 1
      IF (ABS(Y2-Y3).GE.ACC) GO TO 1
      XO(1) = (X1(1)+X2(1)+X3(1))/3.0
      XO(2) = (X1(2)+X2(2)+X3(2))/3.0
      YO = OBJ(XO(1),XO(2))
      WRITE (6,9005) C(8),XO(1),XO(2),YO
 9005 FORMAT (7H FOR N=,F7.4,'OPTIMAL VALUES ARE '/3H U=, E8.4,5X,
     1 3H V=,E8.4,5X,10H  K(U,V) =,E8.4  )
      BO=BB(XO(1),XO(2))
```

```
      WRITE(6,9100) BO
9100  FORMAT(' RATIO OF TOTAL BACKORDERS TO TOTAL DEMAND DURING STOCK'/
     1' OUT PERIOD=',F9.7 )
      S= XO(1)-XO(2)
      Q =XO(2)+S*BO
      WRITE(6,3030) Q,S
3030  FORMAT(' HENCE         Q =',E8.4,   ' AND  S=',E8.4)
      WRITE (6,9000) ITER
9000  FORMAT( ' TOTAL NO. OF ITERATIONS PERFORMED TO GET OPTIMAL SOLN.
     1=',I4/)
      GO TO 3000
9006  END
```

APPENDIX II

COMPUTER OUTPUT FOR EXAMPLE 2.8

```
FOR N=  .0100 OPTIMAL VALUES ARE
U=.1979+02        V=.1976+02        K(U,V) =.9994+02
RATIO OF TOTAL BACKORDERS TO TOTAL DEMAND DURING STOCK
OUT PERIOD= .9923894
HENCE        Q =.1979+02 AND S=.3060-01
TOTAL NO. OF ITERATIONS PERFORMED TO GET OPTIMAL SOLN.  =  12

FOR N=  .0500 OPTIMAL VALUES ARE
U=.2046+02        V=.2020+02        K(U,V) =.9965+02
RATIO OF TOTAL BACKORDERS TO TOTAL DEMAND DURING STOCK
OUT PERIOD= .9868439
HENCE        Q =.2046+02 AND  S=.2655+00
TOTAL NO. OF ITERATIONS PERFORMED TO GET OPTIMAL SOLN.  =  10

FOR N=  .1000 OPTIMAL VALUES ARE
U=.2076+02        V=.2023+02        K(U,V) =.9935+02
RATIO OF TOTAL BACKORDERS TO TOTAL DEMAND DURING STOCK
OUT PERIOD= .9868499
HENCE        Q =.2075+02 AND  S=.5307+00
TOTAL NO. OF ITERATIONS PERFORMED TO GET OPTIMAL SOLN.  =   9

FOR N=  .5000 OPTIMAL VALUES ARE
U=.2100+02        V=.1937+02        K(U,V) =.9753+02
RATIO OF TOTAL BACKORDERS TO TOTAL DEMAND DURING STOCK
OUT PERIOD= .9919086
HENCE        Q =.2099+02 AND  S=.1627+01
TOTAL NO. OF ITERATIONS PERFORMED TO GET OPTIMAL SOLN.  =   8

FOR N= 1.0000 OPTIMAL VALUES ARE
U=.2208+02        V=.1949+02        K(U,V) =.9631+02
RATIO OF TOTAL BACKORDERS TO TOTAL DEMAND DURING STOCK
OUT PERIOD= .9935603
HENCE        Q =.2206+02 AND  S=.2587+01
TOTAL NO. OF ITERATIONS PERFORMED TO GET OPTIMAL SOLN.  =   9

FOR N= 2.0000 OPTIMAL VALUES ARE
U=.2276+02        V=.1900+02        K(U,V) =.9514+02
RATIO OF TOTAL BACKORDERS TO TOTAL DEMAND DURING STOCK
OUT PERIOD= .9953157
HENCE        Q =.2274+02 AND  S=.3760+01
TOTAL NO. OF ITERATIONS PERFORMED TO GET OPTIMAL SOLN.  =   8
```

```
FOR N= 3.0000 OPTIMAL VALUES ARE
U=.2295+02        V=.1892+02        K(U,V) =.9453+02
RATIO OF TOTAL BACKORDERS TO TOTAL DEMAND DURING STOCK
OUT PERIOD=.9966531
HENCE         Q =.2294+02 AND  S=.4026+01
TOTAL NO. OF ITERATIONS PERFORMED TO GET OPTIMAL SOLN.     =  5

FOR N= 4.0000 OPTIMAL VALUES ARE
U=.2276+02        V=.1861+02        K(U,V) =.9418+02
RATIO OF TOTAL BACKORDERS TO TOTAL DEMAND DURING STOCK
OUT PERIOD= .9974106
HENCE         Q =.2275+02 AND S=.4151+01
TOTAL NO. OF ITERATIONS PERFORMED TO GET OPTIMAL SOLN.     =  8

FOR N= 5.0000 OPTIMAL VALUES ARE
U=.2310+02        V=.1881+02        K(U,V) =.9394+02
RATIO OF TOTAL BACKORDERS TO TOTAL DEMAND DURING STOCK
OUT PERIOD=.9978614
HENCE         Q =.2309+02 AND S=.4289+02
TOTAL NO. OF ITERATIONS PERFORMED TO GET OPTIMAL SOLN.     =  8

FOR N= 6.0000 OPTIMAL VALUES ARE
U=.2291+02        V=.1844+02        K(U,V) =.9379+02
RATIO OF TOTAL BACKORDERS TO TOTAL DEMAND DURING STOCK
OUT PERIOD= .9981422
HENCE         Q =.2290+02 AND S=.4470+02
TOTAL NO. OF ITERATIONS PERFORMED TO GET OPTIMAL SOLN.     =  7

FOR N=7.0000 OPTIMAL VALUES ARE
U=.2291+02       V=.1844+02         K(U,V) =.9367+02
RATIO OF TOTAL BACKORDERS TO TOTAL DEMAND DURING STOCK
OUT PERIOD= .9984109
HENCE         Q =.2290+02 AND S=.4470+01
TOTAL NO. OF ITERATIONS PERFORMED TO GET OPTIMAL SOLN.     =  7
```

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